Influences, traces, tribes and thresholds: new examples and some conjectures

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Based on recent (2012) joint work with Jeff Kahn, where we disproved an old conjecture by Benny Chor (1989, unpublished). I will also mention another joint work with Kahn (2006). The lecture also discusses an older joint work with Saharon Shelah (2003, unpublished), a work by Bollobas and Radcliffe (1995), and ancient stuff as well.

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Part I: KKL's theorem and Chor's conjecture

The discrete n-dimensional cube and Boolean functions

The **discrete n-dimensional cube** Ω_n is the set of 0-1 vectors of length n. The uniform probability distribution on Ω_n is denoted by μ.

A **Boolean function** f is a map from Ω_n to $\{0, 1\}$. For a Boolean function f, $\mu(f) = \mu(\{x : f(x) = 1\})$.

A boolean function f is **monotone** if f cannot decrease when you switch a coordinate from 0 to 1.

Influence of variables on Boolean functions

Let

$$\sigma_k(x_1,\ldots,x_{k-1},x_k,x_{k+1},\ldots,x_n)=(x_1,\ldots,x_{k-1},1-x_k,x_{k+1},\ldots,x_n).$$

The influence of the kth variable on a Boolean function f is defined by:

$$I_k(f)=\mu(x\in\Omega_n, f(x)\neq f(\sigma_k(x))).$$

Influence of sets of variables on Boolean functions

Following Ben-Or and Linial we define, for a given f and $S \subset [n]$, the influence of S toward one to be

$$I_S^+(f) = \mu_{[n]\backslash S}(\{u \in \Omega([n]\backslash S) : \exists v \in \Omega(S), f(u,v) = 1\}) - \mu(f).$$

Similarly, the influence of S toward zero is

$$I_S^-(f) = \mu_{[n]\backslash S}(\{u \in \Omega([n]\backslash S) : \exists v \in \Omega(S), f(u,v) = 0\}) - (1 - \mu(f)).$$

The (overall) influence of S is

$$I_S(f) = I_S^+(f) + I_S^-(f).$$

Monotonicity and shifting

Theorem: (Shifting) One can replace g by a monotone Boolean function f so that $\mu(f) = \mu(g)$ and

$$I_S^+(f) \leq I_S^+(g),$$

and

$$I_S^-(f) \leq I_S^-(g),$$

for every S.

KKL's theorem

Theorem (Kahn, K, Linial, 1988) There exists a variable k such that

$$I_k(f) \geq C\mu(f)(1-\mu(f)) \log n/n.$$

A sharp version (due to Talagrand)

$$\sum I_k^p(f)/\log(e + I_k^p(f)) \geq C(p)\mu_p(f)(1-\mu_p(f)).$$

KKL's theorems for influence of sets

Theorem For every $b > 0$ there is $c > 0$ such that if $\mu(f) \approx 1/2$ then there exists a set S of bn variables such that

$$I_S(f) \geq 1 - 1/n^c.$$

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Chor's conjecture

Benny Chor conjectured that much more is true:

Conjecture, Chor(around 1989): For every $b > 0$ there is $c > 1$ such that if $\mu(f) \approx 1/2$ then there exist a set S of bn variables such that,

$$I_S(f) \geq 1 - 1/c^n.$$

Another conjecture from the same time was that:

Conjecture (K? L? also around 1989) For every $b > 0$ there is $c < 1$ such that if $\mu(f) = 1/c^n$ then there exist a set S of variables such that, $|S| = bn$ and

$$I_S(f) \geq 0.9.$$

A counterexample to Chor's conjecture

Theorem (Kahn and Kalai 2012)
For any fixed $\delta > 0$ there is a Boolean function f on n variables, such that $\mu(f) \approx 1/2$ and for every set S of $(1-\delta)n$ variables

$$I_S^+(f) \leq (1-\mu(f)) - 1/n^C,$$

for some fixed $C = C_\delta$.

Theorem (Kahn and Kalai 2012) For any fixed $\delta > 0$ there are $\epsilon > 0$, $\alpha > 0$ and a Boolean function f on n variables, such that $\mu(f) \geq (1-\epsilon)^n$ and for every set S of $(1/2-\delta)n$ variables

$$I_S^+(f) \leq exp[-n^\alpha].$$

Part II: Tribes, tribes, tribes 1

Ben-Or Linial tribes example

A tribe example is a function of the form $\vee_{i=1}^{m} C_i$ where each C_i is an $\wedge$ of k variables.

In Ben-Or Linial example, the C_is are disjoint set of variables, $k = \log n - \log\log n + c$ and $m = n/k$.

A dual tribe example is a function of the form $\wedge_{i=1}^{m} C_i$ where each C_i is an $\vee$ of k variables. (Of course dual tribes have the same influences as tribes with signs reversed.)

Harper's theorem vs. KKL

Let f be a Boolean function and $\mu(f) = t$,

$$I(f) := \sum_{k=1}^{n} I_k(f) \geq 2t \log(1/t).$$

Therefore, $\max I_k(f) \geq 2t \log(1/t)/n$.This matches what KKL gives if $t = 1/n^C$. Can KKL be improved then? Let's try tribes.

Modified tribes examples

We take n/k tribes of size k, if $k = \log n - 2\log\log n + c$ then $\mu(f) = 1 - 1/n$. If f represents the dual-tribes function with tribes of this size, we get that $\mu(f) = 1/n$. What is the influence?

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Perhaps KKL can be improved?

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KKL cannot be improved! Harper's inequality gives the right answer for small t, namely when $t = O(1/n^c)$.

The Ajtai-Linial example

BL and KKL asked if when $\mu(f) = 1/2$ is there alway a set S with $|S| = n^\alpha$, $\alpha < 1$ with $I_S^+(f) = 1/2 - o(1)$ or $I_S^-(f) = 1/2 - o(1)$? Miklos Ajtai and Nati Linial showed that you need that S will be as large as $n/\log^2 n$.

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Ideas: 1) Replace variables by literals in the tribe example. (A literal is a variable or its negation.)

2) Take a Boolean circuit of depth 3. ORS of ANDS of ORS.

3) Take a random such example.

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Conjecture (Friedgut, others): This is best possible.

Hastad, Boppana, LMN (Linial, Mansour, Nisan)

If f, a Boolean function, is represented by a depth d size M Boolean circuit then

$$I(f) \leq (\log M)^{d-1}.$$

Conclusion so far

To prove Chor's conjecture (and the other one) we need a method that will directly exploit influences of large sets.

Part III: Traces, Sauer-Shelah's theorem and beyond

Traces

For a set X put $2^X = \{S : S \subset X\}$, $\binom{X}{k} = \{S \subset X : |S| = k\}$, and $\binom{X}{\le k} = \{S \subset X : |S| \le k\}$. Let $X = [n] = \{1, 2, \ldots, n\}$. For a family of sets $\mathcal{F} \subset 2^X$ and a subset $Y \subset X$, the *trace* of $\mathcal{F}$ on Y denoted by $\mathcal{F}_{|Y}$ is defined by

$$\mathcal{F}_{|Y} = \{R \subset Y : R = S \cap Y \text{ for some } \ S \in \mathcal{F}\}.$$

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The Sauer-Shelah theorem describes for every n and m, the minimal cardinality of $\mathcal{F}$ which guarantees the existence of a subset $Y \subset X$, $|Y| = m$, with a "complete trace," namely, such that $\mathcal{F}_{|Y} = 2^Y$. In this case, it is said that Y is *shattered* by the family $\mathcal{F}$.

The Sauer-Shelah Theorem:

Let $\mathcal{F} \subset 2^X$ satisfies

$$|\mathcal{F}| > \binom{n}{\leq m} =: \binom{n}{m} + \binom{n}{m-1} + \cdots \binom{n}{0},$$

then there exists a subset $Y \subset X$, $|Y| = m+1$, such that $\mathcal{F}_{|Y} = 2^Y$.

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The family $\mathcal{F} = \binom{X}{\leq m}$, shows that this result is sharp. This family is referred to as a *Hamming ball*. (It is a ball around the empty set with respect to the Hamming metric on 2^X.)

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The Sauer-Shelah Theorem was proved around the same time by Sauer, Shelah and Perles, and Vapnik and Chervonenkis. It has many connections, applications and extensions in combinatorics, probability theory, model theory, analysis, statistics and other areas.

An arrow notation:

The results and conjectures described so far can be presented in terms of the following arrow notation: $(N, n) \rightarrow (M, k)$ means that for every family of subsets of $[n]$ of size N has a trace of size M or more, on some subset $S \subset [n]$, with $.|S| = k$ The Sauer-Shelah theorem asserts that

$$(\binom{n}{< k} + 1, n) \rightarrow (2^k, k).$$

It is of interest to understand, more generally, when the relation $(N, n) \rightarrow (M, k)$ holds. In particular, It is natural to ask what is the situation if we relax the conclusion "complete trace" to "almost complete trace" which means that $\mathcal{F}_{|Y}$ consists of a large fraction of sets in 2^Y. Is the Hamming ball still optimal?

The connection with influences

The connection between traces and influences is as follows Let f be a Boolean function on Ω_n and $\mathcal{F} = f^{-1}(1)$, considered as a family of subsets of $[n]$.
It is easy to see that for $S \subset [n]$ and $T = [n] \backslash S$,

$$I_S^+(f) + \mu(f) = 2^{-|T|}|\mathcal{F}_{|T}|.$$

Thus, in the language of traces, we are interested in the effect of relaxing "$\mathcal{F}$ shatters Y" to require only that $\mathcal{F}_{|Y}$ contain a large fraction of 2^Y.

Part IV: Bollobas and Radcliffe (1995)

A super bold working conjecture

The following super-bold working-conjecture which implies the bold working-conjecture was considered by Bollobas and Radcliffe (and immediately disproved by them) ,

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The tribe example is a counterexample for $m = n - 1$.

Bollobas and Radcliffe's example

Bollobas and Radcliffe gave a counter example to this conjecture showing that

Theorem: For every $\epsilon > 0$ if n is large enough,

$$(\binom{n}{< k} + 1, n) \not\to (\epsilon n^{k-1}, n/2).$$

Their construction is based on taking a random family of sets of size $\log \log n$ and considering the family of all subsets of these sets.

Part V: Saharon and me

The bold super conjecture

Bold working conjecture: For every $\epsilon > 0$ if m is sufficiently large,

$$(\binom{n}{\leq (1+\epsilon)m/2}, n) \rightarrow ((1-\epsilon)2^m, m).$$

A Counterexample by Kalai-Shelah (around 2003)

We consider a nonnegative integers r and let $k = 2r$, and $m = 3r$. We first choose a random family $\mathcal{G}$ in $\binom{X}{r}$ where each r-set is chosen with a certain probability p. Next we let $\mathcal{F}$ to be the set of k-subsets S of X so that every T, $S \subset T$, $|T| = r$ is a member in $\mathcal{G}$.

This shows that

$$(n^{m-\delta}, n) \not\to ((\epsilon \cdot 2^m, m).$$

Part VI: The new examples with Jeff (or, more tribes)

The construction for Chor

A collection of m sets of size k of variables, and on each set a constraint for each variable. The function f is the associated dual-tribe: you need to satisfy at least one constraint for every set. $k = C \log n$, m is appropiately chosen.

Part VII: So what further can be said?

Is it true that for each fixed $\delta > 0$ there are a fixed β and Boolean functions f such that $\mu(f) > n^{-\beta}$ and no set of size $(1/2 - \delta)n$ has influence to 1 greater than δ?

Guess: No. but yes for $\mu(f) > exp[(\log n)^{\beta}]$.

Part VIII: The expectation threhold conjecture

Consider a random graph G in $G(n, p)$ and the graph property: G contains a copy of a specific graph H. (Note: H depends on n; a motivating examp le: H is a Hamiltonian cycle.) Let q be the minimal value for which the expected number of copies of H' in G is at least $1/2$ for every subgraph H' of H. Let p be the value for which the probability that G contains a copy of H is $1/2$.

Conjecture: [Kahn, K. 2006]

$$p/q = O(\log n).$$

The conjecture can be vastly extended to general Boolean functions. Connectivity shows: the $\log n$ factor cannot be removed. Note: connectivity is essentially a dual-tribe example.

Pseudo-juntas and a super-Hatami conjecture

Suppose that for every subset of variables S, we have a function $J_S : \{0,1\}^S -> \{0,1\}$ which can be viewed as a constraint over the variables with indices in S. Now there are two conditions:
A Boolean function is a K-**psudo-junta** if
(1) the expected number of variables in satisfied constraints is bounded by K.
(2) $f(x) = f(y)$ if the variables in satisfied constraints and also their values are the same for x and y.

Super - Hatami's conjecture (uniform case): For every C there is $K(C)$, such that if

$$I(f) < Ct \log(1/t),$$

then f is close to a $K(C)t \log(1/t)$-pseudo-junta.
The super-Hatami conjecture and other conjectures by Jeff and me regarding expectation threshold, co-exist but are rather in the opposite direction of our new constructions.

Part IX: Robustness of gradual density increase methods for other problems?

Other examples of the gradual-density increase method.

Roth's theorem and Meshulam-Roth theorem

Density Hales-Jewett theorem