Threshold Behavior of Random Structures - Advances and Challenges

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Erdős and Rényi

The critical probability for connectivity for $G(n, p)$ is $\log n/n$ the threshold interval for connectivity is of magnitude $1/n$. 
Erdős and Rényi
For every $\epsilon > 0$ there exists $k$ such that for every monotone (graph) property $A$, if the probability that $G \in G(n, p)$ satisfies $A$ is at least $\epsilon$ then the probability that $G \in G(n, kp)$ satisfies $A$ is at least $1 - \epsilon$.

In other words, for fixed $\epsilon > 0$ the length of the threshold interval is bounded by a constant times the critical probability.
Heppy birthday dear Bela!!!
Heppy birthday dear Bela!!!
Four threshold men
This lecture
This lecture is about advances and challenges in the abstract study of threshold behavior of stochastic systems, and relations with discrete isoperimetry and discrete harmonic analysis. The abstract study of threshold behavior is a small but interesting fragment of the wide theory of threshold behavior for random graphs and related stochastic properties.

I will start by mentioning one major theorem (Friedgut 1999) and one conjecture (Kahn and Kalai 2006).
Introduction: A theorem and a conjecture
Friedgut’s theorem

Definition: A $B$-local monotone property of graphs is the property of containing a graph among a finite family of graphs of bounded size $B$.
Example: To contain a $K_5$ is a local property.

Notation: For a monotone property of graphs $P$ denote by $\mu_p(P)$ the probability that a random graph in $G(n,p)$ satisfies $P$.

Friedgut’s theorem (1999): For every $\epsilon$ and $C$, there is $B$ such that for every monotone property $P$ of graphs, if

$$d \mu_p(P)/dp < pC,$$

then $P$ is $\epsilon$-close to a $B$-local monotone property $Q$. 
A striking consequence of Friedgut’s theorem

Coarse threshold occurs only at threshold functions of the form $n^\beta$ where $\beta$ is rational.

**Corollary:** Connectivity has a sharp threshold. (Because the threshold function is $\log n/n$.)
Threshold and Expectation threshold

Consider a random graph $G$ in $G(n, p)$ and the graph property: $G$ contains a copy of a specific graph $H$. (Note: $H$ depends on $n$; a motivating example: $H$ is a Hamiltonian cycle.) Let $q$ be the minimal value for which the expected number of copies of $H'$ in $G$ is at least $1/2$ for every subgraph $H'$ of $H$. Let $p$ be the value for which the probability that $G$ contains a copy of $H$ is $1/2$.

**Conjecture:** [Kahn, Kalai 2006]

$$\frac{p}{q} = O(\log n).$$

The conjecture can be vastly extended to general Boolean functions, and we will hint on possible connection with harmonic analysis and discrete isoperimetry. (Sneak preview: it will require a far-reaching extension of results by Friedgut, Bourgain and Hatami.)
Part II: Boolean functions and influences
The discrete $n$-dimensional cube and Boolean functions

The **discrete $n$-dimensional cube** $\Omega_n$ is the set of 0-1 vectors of length $n$.

A **Boolean function** $f$ is a map from $\Omega_n$ to $\{0, 1\}$.

A boolean function $f$ is **monotone** if $f$ cannot decrease when you switch a coordinate from 0 to 1.
The Bernoulli measure

Let $p$, $0 < p < 1$, be a real number. The probability measure $\mu_p$ is the product probability distribution whose marginals are given by $\mu_p(x_k = 1) = p$. Let $f : \Omega_n \rightarrow \{0, 1\}$ be a Boolean function.

$$\mu_p(f) = \sum_{x \in \Omega_n} \mu_p(x)f(x) = \mu_p\{x : f(x) = 1\}.$$
The total influence

Two vectors in $\Omega_n$ are **neighbors** if they differ in one coordinate.

For $x \in \Omega_n$ let $h(x)$ be the number of neighbors $y$ of $x$ such that $f(y) \neq f(x)$.

The **total influence** of $f$ is defined by

$$I^p(f) = \sum_{x \in \Omega_n} \mu_p(x)h(x).$$

If $p = 1/2$ we will omit $p$ as a subscript or superscript.
Russo’s lemma

**Russo’s lemma:** For a monotone Boolean function $f$, 

$$d\mu_p(f)/dp = I^p(f).$$

Very useful in percolation theory and other areas.

The **threshold interval** for a monotone Boolean function $f$ is those values of $p$ so that $\mu_p(f)$ is bounded away from 0 and 1. (Say $0.01 \leq \mu_p(f) \leq 0.99$.)

A typical application of Russo’s lemma: If for every value $p$ in the threshold interval $I^p(f)$ is large, then the threshold interval itself is short. This is called a **sharp threshold** phenomenon.
(A version of) Harper’s theorem

Harper’s theorem: If $\mu_p(f) = t$ then

$$\|p(f) \| \geq 2t \cdot \log_p t.$$ 

There is a 3 line proof by induction.

Harmonic analysis proof: without the $\log(1/t)$ factor it follows from Parseval.

Gil Kalai
Thresholds
Influence of variables on Boolean functions

Let

\[ \sigma_k(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) = (x_1, \ldots, x_{k-1}, 1-x_k, x_{k+1}, \ldots, x_n). \]

The influence of the \( k \)th variable on a Boolean function \( f \) is defined by:

\[ I_k^p(f) = \mu_p(x \in \Omega_n, f(x) \neq f(\sigma_k(x))). \]
KKL’s theorem

**Theorem** (Kahn, K, Linial, 1988; Bourgain Katzenelson KKL 1992; Talagrand 1994 Friedgut Kalai 1996) There exist a variable $k$ such that

$$I_k(f) \geq C \mu(f) (1 - \mu(f)) \log n/n.$$ 

A sharp version (due to Talagrand)

$$\sum I_k^p(f)/\log(1/I_k^p(f)) \geq C(p)\mu_p(f)(1 - \mu_p(f)).$$
Discrete Fourier analysis

We assume now $p = 1/2$. Let $f : \Omega_n \to \mathbb{R}$ be a real function. Let $f = \sum \hat{f}(S)W_S$ be the **Fourier-Walsh** expansion of $f$. Here

$$W_S(x_1, x_2, \ldots, x_n) = (-1)^{\sum\{x_i : i \in S\}}.$$
Hypercontractivity and Harper’s theorem:

We assume now $p = 1/2$. $f = \sum \hat{f}(S) W_S$ is the Fourier-Walsh expansion of $f$. Key ideas:

0 Parseval gives $I(f) = 4 \sum \hat{f}^2(S)|S|$.

1 Bonami-Gross-Beckner hypercontractive inequality.

$$\| \sum \hat{f}(S)(1/2)^{|S|} \|_2 \leq \| f \|_{5/4}.$$  

2 For Boolean functions the $q$th power of the $q$ norm is the measure of the support and does not depend on $q$. If the support is small this means that the $q$-norm is very different from the $r$-norm if $r \neq q$.

(See also : Ledoux’ book on concentration of measure phenomena)
Part III: Thresholds: Baker Dozen’s challenges

1. **Threshold and symmetry.** The threshold window for “containing a clique of size $\log n$” is of size $1/\log^2 n$. This seems to be the largest threshold for graph properties. Where does the extra $\log n$ come from?

2. **Connectivity; small $p$s.** Understand the sharp threshold behavior of connectivity as part of a general framework. More generally, understand threshold intervals when $p$ is small.

3. **Hunting sharp thresholds.** Prove sharp-threshold behavior for various monotone properties of random graphs.

4. **Uniform vs. non-uniform models.** Understand the threshold behavior for the 3-SAT problem, 3-colorability of graphs, etc.. Relate to 0-1 laws.
5 More isoperimetry. Margulis and Talagrand extended the Erdős-Rényi result on connectivity in a different direction, relying on a different isoperimetric result. Combine this study with the study of influence and discrete Fourier analysis.

6 General conditions for sharp threshold. What is a necessary and sufficient condition for a sequence of Boolean functions to have the sharp threshold property?

7 Locating the threshold. Can one sometimes find and prove the location of the threshold based on sharp threshold behavior?

8 Larger alphabets, etc. Study larger alphabets, products of other groups, products of graphs, analogous settings for the symmetric group.

9 Positional games and other games. Extend the influence/sharp-threshold framework to positional games on graphs and other games.

10 Models of statistical physics. Apply the influence/shreshold theory to percolation and other models of statistical physics.
Polynomially-small threshold intervals. Are there general conditions that will guarantee that the total influence is at least $n^\alpha$ for some $\alpha > 0$? At most $n^{1/2-\alpha}$ for some $\alpha > 0$?

Phase transition phenomena. Relations between influence, isoperimetry and discrete Fourier analysis and other phase transition phenomena for random graph such as the emergence of giant component.

Non-product distributions. Such as the equal-slice distribution, distributions arising in the Potts model, FKG distributions.

Graph parameters rather than properties. Study distributions, concentration phenomena, etc. for parameters of random graphs.

Continuous settings. Extend the notions and the results to functions with continuous domain and range. Important examples: Gaussian spaces, Gelfand pairs,...
Part IV: Threshold behavior and symmetry
Invariance under transitive group

**Theorem:** If a monotone Boolean function $f$ with $n$ variables is invariant under a transitive group of permutation of the variables, then

$$I^p(f) \geq C \mu_p(f)(1 - \mu_p(f)) \log n.$$ 

**Proof:** Follows from KKL’s theorem (extended to general Bernoulli measures) since all individual influences are the same.
Total influence under symmetry of primitive groups

For a transitive group of permutations $\Gamma \subset S_n$, let $I(\Gamma)$ be the minimum influence for a $\Gamma$-invariant function Boolean function with $n$ variables.

**Theorem:** [Bourgain and Kalai 1998] If $\Gamma$ is primitive then one of the following possibilities hold.

- $I(\Gamma) = \theta(\sqrt{n})$,

- $(\log n)^{(k+1)/k-o(1)} \leq I(\Gamma) \leq C(\log n)^{(k+1)/k}$,

- $I(\Gamma)$ behaves like $(\log n)\mu(n)$, where $\mu(n) \leq \log \log n$ is growing in an arbitrary way.
Jumps in the behavior of $I(\Gamma)$ for primitive groups $\Gamma$

If $\Gamma$ is not $A_n$ and $S_n$ then $I(\Gamma) \leq (\log n)^2$.

If $I(\Gamma) \leq (\log n)^{1.99}$ then $I(\Gamma) \leq (\log n)^{3/2}$

If $I(\Gamma) \leq (\log n)^{3/2-\epsilon}$ then $I(\Gamma) \leq (\log n)^{4/3}$

If $I(\Gamma) \leq (\log n)^{4/3-\epsilon}$ then $I(\Gamma) \leq (\log n)^{5/4}$

...\n
If $I(\Gamma) \leq (\log n)^{1+o(1)}$ then $I(\Gamma) \leq \log n \cdot \log \log n$
Threshold behavior for random graphs

The case that $\Gamma$ is $S_n$ acting on unordered pairs from $[n] = \{1, 2, \ldots, n\}$ describes graph properties. The conclusion is that the threshold interval for graph properties is at most

$$1/ \log^{2-o(1)} n.$$
Hypercontractivity and the lower bounds

Both the upper bounds and the lower bounds depend on finding invariants of the group which causes the threshold to go above $\log n$. Giving constructions for the upper bounds requires a detailed understanding of primitive permutation groups based on the classification theorem and O’Nan-Scott theorem.

The lower bounds are based on delicate and complicated harmonic analysis.

**Step I:** hypercontractivity $+$ random restriction argument $+$ clever inequalities takes you in the graph case from $\log n$ to $\log n^{3/2}$.

**Step II:** Extremely subtle ”bootstrap” to amplify the outcome.
The Entropy Influence conjecture (Friedgut + Kalai 1996)

If the Fourier-Walsh expansion of \( f \) is 
\[
f = \sum \hat{f}(S) W_S
\]
define
\[
E(f) = \sum \hat{f}^2(S) \log(1/\hat{f}^2(S)).
\]

**Conjecture:** For some absolute constant \( C \),
\[
I(f) \geq C \cdot E(f).
\]
Scaling-limit symmetry, critical exponents, spectral distribution,...
Prelude: A necessary and sufficient condition for $o(1)$ threshold window.

The Shapley value of the $k$th variable is defined by

$$
\psi_k(f) = \int_0^1 l_k^p(f) dp.
$$

**Theorem:** (Kalai 2005) A necessary and sufficient condition for diminishing threshold window is that the maximum of the Shapley values tends to 0.

**Problem 1:** Close the exponential gap in this theorem.

**Problem 2:** Find a necessary and sufficient condition for sharp threshold (when the critical probability is small).
Part V: Stability for edge-isoperimetry: From Friedgut to Hatami and beyond
Low Influence and Juntas

A **dictatorship** is a Boolean function depending on one variable. A **$K$-junta** is a Boolean function depending on $K$ variables.

**Theorem:** (Friedgut) If $p$ is bounded away from 0 and 1 and $I^p(f) < C$ then $f$ is close to a $K(C)$-Junta.

This works if $\log p / \log n = o(1)$ the most interesting applications would be when $p$ is a power of $n$. There the theorem is not true.
The works of Friedgut and Bourgain (1999)

Suppose that $f$ is a Boolean function and

$$l^p(f) < pC,$$

then

**Friedgut’s theorem** (1999) (already mentioned): If $f$ represent a monotone graph property then $f$ is close to a “locally defined” monotone function $g$.

**Bourgain’s theorem** (1999): Unconditionally, $f$ has a substantial “locally defined” ingredient.
The hunt for sharp thresholds

These results are very useful for proving sharp threshold for specific examples. However implementing them often requires difficult analysis. Here are a few examples where sharp thresholds were established.

- K-SAT (Friedgut)
- Colorability (Achlioptas, Friedgut)
- Ramsey properties for graphs (Friedgut Krivelevich; Friedgut, Rodl, Rucinski, Tetali)
- Random sets with a monochromatic arithmetic progression in every 2-coloring (Friedgut, Han, Person, Schacht)
- $k$-cores (Shoham Letzter).
Hatami’s theorem: Pseudo-juntas

Suppose that for every subset of variables $S$, we have a function $J_S : \{0, 1\}^S \rightarrow \{0, 1\}$ which can be viewed as a constraint over the variables with indices in $S$. Now there are two conditions:

A Boolean function is a $K$-pseudo-junta if

1. the expected number of variables in satisfied constraints is bounded by a constant $K$.
2. $f(x) = f(y)$ if the variables in satisfied constraints and also their values are the same for $x$ and $y$.

**Hatami’s theorem:** For every $C$ there is $K(C)$, such that if

$$\mathbb{I}_p(f) < pC,$$

then $f$ is close to a $K(C)$-pseudo-junta.
A conjectural extension of Hatami’s theorem

**Conjecture:** Suppose that \( \mu_p(f) = t \) and

\[ I(f) \leq C \log(1/t)t \]

then \( f \) is \( \epsilon t \)-close to a \( O(\log(1/t)) \)-pseudo-junta.

If true, may apply toward the conjecture on expectation thresholds, and may enable us to use the abstract threshold theory for hunting thresholds’ locations.
Thank you very much!