Hypergraphs and Geometry

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Thm [Frankl-Wilson(81)]: If $p$ is prime, $F$ a family of $r$-subsets of an $n$-set, and

$$|A \cap B| \equiv \lambda_1, \lambda_2, \ldots, \text{or } \lambda_s \mod p$$

for any two distinct $A, B$ in $F$, where

$$\lambda_i \neq r \mod p$$

for any $i$, then

$$|F| \cdot \binom{n}{s}$$

Corollary 1 (FW): The chromatic number of the unit distance graph in $\mathbb{R}^n$ is exponential in $n$.

This settles a question of Erdős and another of Larman and Rogers
More Corollaries:

**Corollary 2 [Kahn-Kalai (93)]:** there are (finite) subsets of $\mathbb{R}^n$ that cannot be partitioned into less than $1.1^n$ sets of smaller diameter.

This settles a question of Borsuk

**Corollary 3 [A (98)]:** there are graphs $G$ and $H$ so that the Shannon capacity of their disjoint union is bigger than the sum (or any fixed power of the sum) of their capacities.

This settles a question of Shannon.
II. Zoltan

P. Erdős and Z. Füredi, The greatest angle among n points in the d-dimensional Euclidean space, 1981.
Thm [Erdős-Füredi (81)]: The maximum cardinality of a set of points in $\mathbb{R}^d$ which determines only acute angle triangles is exponential in $d$.

This settles a problem of Danzer and Grünbaum.

The proof is closely related to:

P. Erdős, P. Frankl, Z. Füredi, families of finite sets in which no set is covered by the union of two others, JCT(A), 1982.
N. Alon and E. Győri, The number of small semi spaces of a finite set of points in the plane, JCT(A), 1986.
A **k-set** of a set $S$ of $n$ points in the plane is a $k$-subset of $S$ which is the intersection of $S$ with a half-plane (here $k \leq n/2$).

**Erdős, Lovász, Simmons and Straus (75):** The maximum possible number of $k$-sets of a planar $n$-set is at least $\Omega(n \log k)$ and at most $O(n k^{1/2})$.

**Pach, Steiger and Szemerédi (89):** at most $O(\ nk^{1/2} / \log^* k)$

**Dey (97):** at most $O(nk^{1/3})$

**Toth(99):** at least $n^2 \left(\sqrt{\log k}\right)$
Thm [A-Győri (86)]: The maximum possible value of the number of 1-sets+2-sets+ …+k-sets in a planar set of n points is exactly \( kn \) (for all \( k \leq n/2 \)).

This can be proved geometrically, or using the method of allowable sequences of Goodman and Pollack.
János

For \( u,v \) in \( \mathbb{R}^d \), \( \text{Box}(u,v) \) is the smallest axis parallel box containing \( u \) and \( v \).

**Bárány and Lehel (87):** For any \( d \) there is \( f \) so that any finite or compact \( S \) in \( \mathbb{R}^d \), contains a subset \( X \) of at most \( f(d) \) points so that \( S \) is contained in the union of the boxes \( \text{Box}(u,v) \) with \( u,v \) in \( X \).
Proof gives: \( f(d) \leq \left( 2d^{2d} + 1 \right)^{2d} \)

**Pach (97):** \( f(d) \leq 2^{2d+2} \)

Proof is based on a relation, proved by Ding, Seymour and Winkler (94), between the cover number and matching number of a hypergraph containing no k edges with each pair sharing a vertex not contained in any of the other k-2 edges.

**Known:** \( f(d) \geq 2^{2d-1} \)
A, Brightwell, Kierstead, Kostochka, Winkler (06):

\[ f(d) \cdot 2^{2^d} + d + \log d + \log \log d + O(1) \]

Proof is based on the relation between the fractional and integral cover numbers of a hypergraph with a bounded VC dimension [Komlós, Pach, Woeginger (92), following Haussler, Welzl (87) and Vapnik, Chervonenkis (71)]
Strange (hyper)graphs
(constructed geometrically)
Problem (Erdős-Rothschild): Determine or estimate $h(n,c)=\text{maximum } h \text{ so that any graph with } n \text{ vertices and at least } cn^2 \text{ edges in which every edge is contained in a triangle must contain an edge lying in at least } h(n,c) \text{ triangles.}$

Fact (Szemerédi): for any fixed $c>0$, $h(n,c)$ tends to infinity with $n$, in fact, it is at least a power of $\log^* n$ [Fox: at least exponential in $\log^* n$]

Fox and Loh (13): For any fixed $c<1/4$, $h(n,c) \leq n^{O(1/ \log \log n)} (= n^{o(1)})$. 
A related result [A, Moitra, Sudakov (13)]:
There are graphs on n vertices with
\[
(1 - o(1)) \binom{n}{2}
\]
edges, that can be decomposed into pairwise edge disjoint \textbf{induced matchings}, each of size
\[n^{1-o(1)}\]

That is: there are \textbf{nearly complete} graphs, which can be decomposed into pairwise edge disjoint
\textbf{nearly perfect induced matchings}.

Note: the existence of such graphs for matchings of large constant size is a special case of a result of Frankl and Füredi (87).
Extension to Hypergraphs

$K = K_4^{(3)} = \text{the complete 3-graph on 4 vertices}$

Let $h_3(n,c)$ denote the maximum integer so that any $n$-vertex 3-graph with at least $cn^3$ edges, each contained in a copy of $K$, must contain an edge lying in a least $h_3(n,c)$ copies of $K$.

By the hypegraph removal lemma, for any fixed $c > 0$, $h_3(n,c)$ tends to infinity with $n$. 
Let $\text{ex}_3(n,K)$ denote the maximum possible number of edges in a 3-graph with no copy of $K=K_4^{(3)}$. The determination of this number is an old problem of Turán, and the limit of the ratio $\text{ex}_3(n,K)/n^3$, which exists by Katona, Nemetz, Simonovits(64), is also open, and conjectured to be $5/9$. Let $d$ denote this limit (the Turán density of $K$).

The results on supersaturated hypergraphs [Erdős, Simonovits(83)] imply that for $c>d(K)$, $h_3(n,c)=\Omega(n)$. 
**Thm 1:** for any fixed $c < d(K)$, $h_3(n, c) \leq n^{O(1/\log \log n)}$.

A related result is

**Thm 2:** There are 3-graphs on $n$ vertices with

$$\left(1 - o(1)\right) \binom{n}{3}$$

edges, whose edges can be decomposed into $n^{1+o(1)}$ induced hypergraphs, each being a partial Steiner Triple System.

That is: there are nearly complete 3-graphs that can be decomposed into pairwise disjoint nearly complete partial Steiner Triple systems.
The proofs use a geometric construction, based on measure concentration in high dimensional Euclidean spaces.
Some proof ideas (sketch):

Construction of nearly complete 3-graphs decomposable into nearly perfect partial Steiner triple systems.

C, d: large integers, \([C]=\{1,2,\ldots,c\}\), \(V=[C]^d\)

\(n=C^d\)

View \(V\) as a set of points in the Euclidean space \(\mathbb{R}^d\).

\(\mu = E \left( \|x - y\|_2^2 \right) \) as \(x,y\) are random members of \(V\).
Fact 1: $\mu = \Theta(C^2d)$ ($>d$)

$H=H(C,d)$ is the 3-graph on $V$ in which $\{x,y,z\}$ forms an edge iff

$$\mu - d \leq ||x-y||^2, ||x-z||^2, ||y-z||^2 \leq \mu + d$$

Fact 2: The number of edges of $H$ is at least

$$\left(1 - 6e^{-d/(2C^4)}\right)n^3/6 \ (> (1 - o(1))\binom{n}{3}$$

provided $d >> C^4$. 
For each vector $w$ in $\{1, 4/3, 5/3, 2, \ldots, C-2/3, C-1/3, C\}^d$ define

$$V_w = \{v \in V : \frac{\mu}{3} - \frac{5d}{9} \leq \|v - w\|^2 \leq \frac{\mu}{3} + \frac{5d}{9}\}$$

Let $H_w$ be the induced sub-hypergraph of $H$ on $V_w$

**Fact 3:** Each edge $\{x, y, z\}$ of $H$ lies in at least one of the hypergraphs $H_w$

**Proof:** take $w = \frac{x+y+z}{3}$ and apply an appropriate geometric inequality
Fact 4: For every pair of vertices \( x, y \) in \( V_w \), the number of edges of \( H_w \) that contain the pair \( v,w \) is at most \( 15^d \) (which is \( n^{o(1)} \) if \( C >> 1 \)).

Proof: A volume argument
Therefore, the edges of each $H_w$ can be partitioned into less than $3 \cdot 15^d$ induced partial Steiner triple systems.

Altogether, the number of these systems is less than $3^d \cdot 3 \cdot 15^d \cdot n$, which is $n^{1+o(1)}$ for $C>>1$.

This proves Theorem 2.
To get Theorem 1 we need a dense 3-graph in which every edge lies in a small positive number of copies of $K=K_4^{(3)}$.

Take $H(C,d)$, add a new vertex $z_w$ for each $H_w$ and let $\{x,y,z_w\}$ be a 3-edge for each $x,y$ in $V_w$.

Then replace the edges of $H(C,d)$ by the intersection of $H(C,d)$ with an extremal $K$-free 3-graph on the same set of vertices and blow-up the vertices of $H(C,d)$ (but not the new vertices $z_w$).
Open :

Fix $c < d(K)$, $h_3(n, c) = ?$