Definitions

- **Strongly regular** $(d - 1)$-sphere:
  a pure, finite, regular CW complex with the intersection property; the underlying space is the $(d - 1)$-sphere $S^{d-1}$.

- **Eulerian lattice**:
  a graded partially ordered set (poset) with least and maximal element, s.t. every interval satisfies the Euler equation and that any two elements have a join and a meet.

- **Flag-vector**:
  The sets of flag vectors are $F(P^d)$, $F(S^{d-1})$, resp. $F(E_{L^{d+1}})$; the flag vector is indexed by $S \subset \{0, ..., d - 1\}$, where $f_k := |\{F_i \subset \cdots \subset F_k \subseteq P \mid \dim(F_i), ..., \dim(F_k) = S\}|$.

Motivation

- What is the space $F(P^d)$ of flag-vectors of $d$-polytopes? Steinitz (1906): $F(P^d) \cong \{(f_0, f_1) \mid f_0 \leq 2f_2 - 4, f_2 \leq 2f_0 - 4\}$.
- g-Theorem [1]: Gives complete description of $f$-vectors of simplicial $d$-polytopes.

- Do spheres and Eulerian lattices have the same or strictly larger sets of flag-vectors $F(S^{d-1})$ resp. $F(E_{L^{d+1}})$?
- g-Conjecture: The conditions of the g-Theorem hold for strongly regular $(d - 1)$-spheres.

The Lower Bound Theorem

- The LBT for simplicial $d$-polytopes extends to $(d - 1)$-spheres and even to $(d - 1)$-dimensional pseudomanifolds (Tay, ‘95).
- In the simplicial case this implies: $F(P^d) = F(S^{d-1})$, for $d = 4, 5$.

Theorem 1 (B., 2014+). Simplicial strongly connected Eulerian lattices are pseudomanifolds.

Corollary 2. In the simplicial case for $d = 4, 5$, we have $F(P^d) = F(S^{d-1}) = F(E_{L^{d+1}})$.

- What about $d \geq 6$?
For $d = 2k + 2$ there are $(d - 1)$-dimensional tori $S^{2k} \times S^1$ that violate the conditions of the g-Theorem (B., ‘14+). Their face lattices are strongly connected Eulerian lattices and can be extended to the odd-dimensional cases.

Corollary 3 (B., ‘14+). In the simplicial case for $d \geq 6$, we have $F(P^d) \neq F(E_{L^{d+1}})$.

- The non-simplicial case
Generalization of the LBT (Kalai, ‘87) to $d$-polytopes:
\[ f_1(P) + \sum_{k \geq 2} (k - 3)f_1^2(P) \geq df_0(P) - \binom{d+1}{2}, \]
where $f_1^k(P)$ denotes the number of $k$-gonal 2-faces of $P$.

Dimension $d = 4$

- Two parameters: fatness and complexity [2]
\[
F := \frac{f_1 + f_2 - 20}{f_0 + f_1 - 10}, \quad C := \frac{f_0 - 20}{f_0 + f_1 - 10}
\]
Equation (1) $\iff C \geq 3; \quad C \geq 3 \Rightarrow F \geq \frac{2}{3}\]

$A$ thin and a fat lattice.

- $d$-Polytopes have a fatness $F \geq \frac{2}{3}$, whereas no upper bound is known. The largest known fatness is $9 - \varepsilon$.

- $b$-Spheres can have arbitrary fatness, whereas finding a lower bound is an open question (in the non-simplicial case).

Theorem 4 (B., 2013+). Let $S$ be a strongly regular 3-sphere.

(i) If $f_{03} \leq 2f_2$, then $F(S) \geq \frac{5}{2}$.
In particular, this holds for 2-simplicial 3-spheres.

(ii) If $S$ is 2-simplicial 2-simple, other than the simplex, then $F(S) \geq 3$.

Lemma 5 (B., 2013+). Let $S$ be a strongly regular 3-sphere with flag-vector $f(S) = (f_0, f_1, f_2, f_0)$.

(i) If $f_{03} \leq 4f_2 + 2$, then $C(S) \geq 3$ (almost simplicial case).

(ii) Let $l \leq f_{03} - 3f_2 = f_0 - f_1 - 3f_3 + f_{10}$ denote the number of non-triangular 2-faces, then
\[2f_0 + f_1 + 10f_3 - 3f_{10} \leq 3f_0 + 7f_1 - 2f_{03} - l \leq 10.

References


Flag Vectors of Polytopes, Spheres and Eulerian Lattices

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