Flag Vectors of Polytopes, Spheres and Eulerian Lattices

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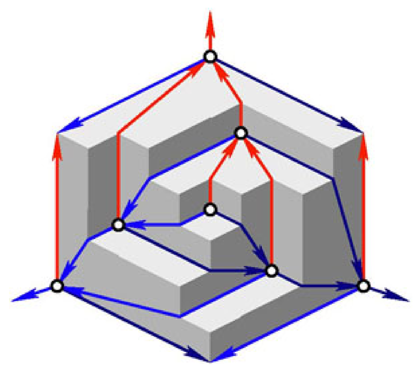


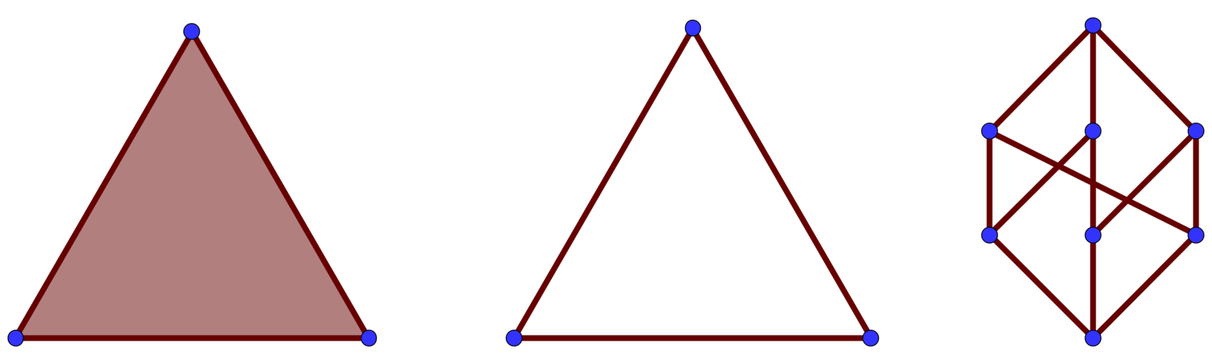
Definitions

- **Strongly regular $(d-1)$-sphere:**
  a pure, finite, regular CW complex with the intersection property; the underlying space is the $(d-1)$-sphere S^{d-1}.
- **Eulerian lattice:**
  a graded partially ordered set (poset) with least and maximal element, s.t. every interval satisfies the Euler equation and that any two elements have a join and a meet.
- **Flag-vector:**
  The sets of flag vectors are $\mathcal{F}(\mathcal{P}^d)$, $\mathcal{F}(\mathcal{S}^{d-1})$, resp. $\mathcal{F}(\mathcal{EL}^{d+1})$; the flag vector is indexed by $S \subset \{0, ..., d-1\}$, where

$$f_S := |\{F_1 \subset \cdots \subset F_k \subseteq P \,|\, \{\dim(F_1), \ldots, \dim(F_k)\} = S\}|.$$

Motivation

- **What is the space $\mathcal{F}(\mathcal{P}^d)$ of flag-vectors of d-polytopes?**
  Steinitz (1906): $\mathcal{F}(\mathcal{P}^3) \cong \{(f_0, f_2) \,|\, f_0 \leq 2f_2 - 4,\ f_2 \leq 2f_0 - 4\}$.
  g-Theorem [1]: Gives complete description of f-vectors of simplicial d-polytopes.
- **Do spheres and Eulerian lattices have the same or strictly larger sets of flag-vectors $\mathcal{F}(\mathcal{S}^{d-1})$ resp. $\mathcal{F}(\mathcal{EL}^{d+1})$?**
  g-Conjecture: The conditions of the g-Theorem hold for strongly regular $(d-1)$-spheres.



A polytope with corresponding sphere and face lattice.

The Lower Bound Theorem

- The LBT for simplicial d-polytopes extends to $(d-1)$-spheres and even to $(d-1)$-dimensional pseudomanifolds (Tay, '95).
  In the simplicial case this implies: $\mathcal{F}(\mathcal{P}^d_s) = \mathcal{F}(\mathcal{S}^{d-1}_s)$, for $d = 4, 5$.

  Theorem 1 (B., 2014+). *Simplicial strongly connected Eulerian lattices are pseudomanifolds.*

  Corollary 2. *In the simplicial case for $d = 4, 5$, we have*

$$\mathcal{F}(\mathcal{P}^d_s) = \mathcal{F}(\mathcal{S}^{d-1}_s) = \mathcal{F}(\mathcal{EL}^{d+1}_s).$$

- **What about $d \geq 6$?**
  For $d = 2k + 2$ there are $(d-1)$-dimensional tori $S^{2k} \times S^1$ that violate the conditions of the g-Theorem (B., '14+). Their face lattices are strongly connected Eulerian lattices and can be extended to the odd-dimensional cases.

  Corollary 3 (B., '14+). *In the simplicial case for $d \geq 6$, we have*

$$\mathcal{F}(\mathcal{P}^d_s) \neq \mathcal{F}(\mathcal{EL}^{d+1}_s).$$

- **The non-simplicial case**
  Generalization of the LBT (Kalai, '87) to d-polytopes:

$$f_1(P) + \sum_{k\geq 3}(k-3)f_2^k(P) \geq df_0(P) - \binom{d+1}{2}, \qquad (1)$$

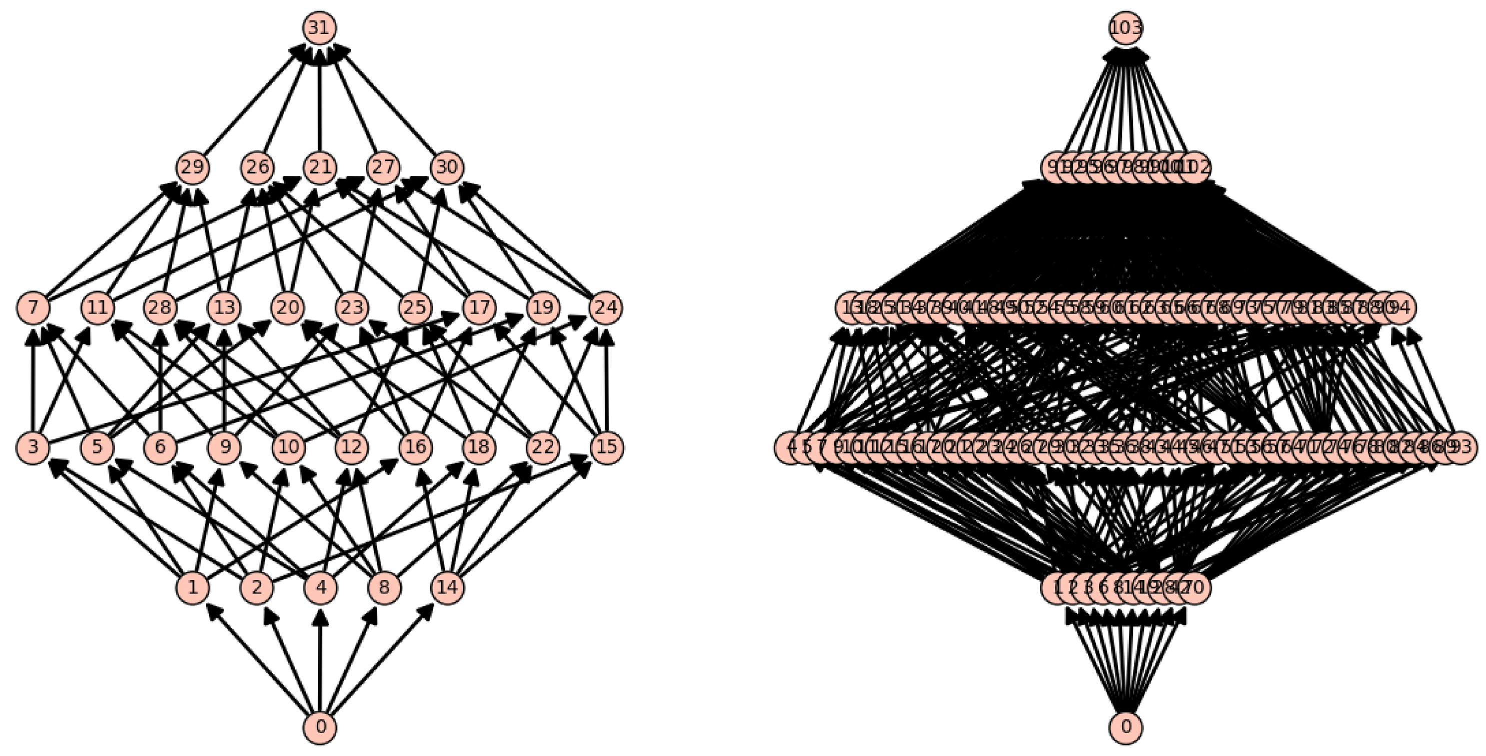
  where $f_2^k(P)$ denotes the number of k-gonal 2-faces of P.

Dimension $d = 4$

- **Two parameters: *fatness* and *complexity* [2]**

$$F := \frac{f_1 + f_2 - 20}{f_0 + f_3 - 10}, \quad C := \frac{f_{03} - 20}{f_0 + f_3 - 10}. \qquad (2)$$

  Equation (1) $\Leftrightarrow$ $C \geq 3$; $\quad C \geq 3 \Rightarrow F \geq \frac{5}{2}$.



A *thin* and a *fat* lattice.

- 4-Polytopes have a fatness $F \geq \frac{5}{2}$, whereas no upper bound is known. The largest known fatness is $9 - \varepsilon$.
- 3-Spheres can have arbitrary fatness, whereas finding a lower bound is an open question (in the non-simplicial case).

  Theorem 4 (B., 2013+). *Let S be a strongly regular 3-sphere.*

  (i) *If $f_{03} \leq 2f_2$, then*

$$F(S) \geq \frac{5}{2}.$$

  In particular, this holds for 2-simplicial 3-spheres.

  (ii) *If S is 2-simplicial 2-simple, other than the simplex, then*

$$F(S) \geq 3.$$

  Lemma 5 (B., 2013+). *Let S be a strongly regular 3-sphere with flag-vector $f(S) = (f_0, f_1, f_3, f_{03})$.*

  (i) *If $f_{03} \leq 4f_3 + 2$, then $C(S) \geq 3$ (almost simplicial case).*

  (ii) *Let $l \leq f_{02} - 3f_2 = f_0 - f_1 - 3f_3 + f_{03}$ denote the number of non-triangular 2-faces, then*

$$2f_0 + f_1 + 10f_3 - 3f_{03} \leq 3f_0 + 7f_3 - 2f_{03} - l \leq 10.$$

References

[1] Edward Swartz. Thirty-Five Years and counting. *arXiv:1411.0987 [math.CO]*, 2014.

[2] Günter M. Ziegler. Convex Polytopes: Extremal Constructions and f-Vector Shapes. In E. Miller and V. Reiner and B. Sturmfels, editor, *"Geometric Combinatorics", Proc. Park City Mathematical Institute (PCMI) 2004*, pages 617–691. American Mathematical Society, Providence, RI, 2007. With an appendix by Th. Schröder and N. Witte.