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On the relations of various conjectures on Latin squares and straightening coefficients

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1. Introduction

We discuss the relations among various combinatorial conjectures, to wit:

I. A conjecture of J. Dinitz on partial Latin squares, closely related to a conjecture of N. Alon and M. Tarsi on Latin squares.

II. A conjecture of the second author on the nonvanishing property of a certain straightening coefficient in the supersymmetric bracket algebra. This conjecture is motivated by the author's program of extending invariant theory by the use of supersymmetric variables.

III. A conjecture of the second author on the exchange property satisfied by sets of bases of a vector space (or more general, for bases of a matroid).

IV. A conjecture of J. Kahn which generalizes Dinitz's conjecture, as well as conjecture III.

We begin by discussing Dinitz's conjecture. Recall that a *partial Latin square* of order n is an $n\times n$ array of symbols with the property that no symbol appears more than once in any row or column. While a graduate student at Ohio State University, Jeff Dinitz proposed the following conjecture.

Conjecture 1 (Dinitz, 1978). Associate to each pair (i,j) where $1\leqslant i,j\leqslant n$ a set S_{ij} of size n. Then there exists a partial Latin square $(a_{ij})_{1\leqslant i,j\leqslant n}$ with $a_{ij}\in S_{ij}$ for all pairs (i,j).

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For example, let $S_{11}=S_{22}=\{a,b\}$, $S_{12}=S_{21}=\{c,d\}$. Then one obtains a partial Latin square of size 2 by choosing $a_{11}=a_{22}=a$, $a_{12}=a_{21}=c$. In this case there are 2^4 different partial Latin squares.

In attempting to solve this conjecture, Alon and Tarsi were led to a new conjecture on Latin squares which implies Dinitz conjecture. Recall that a *Latin square* $(a_{ij})_{1\leqslant i,j\leqslant n}$ is a partial Latin square such that all sets S_{ij} are identical with the set $\{1,2,\ldots,n\}$. A *row inversion* in a Latin square is a pair of numbers in a row which are out of order, that is $a_{ij}>a_{ij'}$ with $j<j'$. A Latin square is *row even* (resp. *row odd*) if it contains an even (resp. odd) number of row inversions. The concepts of column inversions, column even and column odd Latin squares are similarly defined. A Latin square is said to be *even* (resp. *odd*) if the sum of the number of row inversions and the number of column inversions is even (resp. odd). We write RELS(n), ROLS(n), CELS(n), COLS(n), ELS(n) and OLS(n) to denote the number of row even, row odd, column even, column odd, even and odd Latin squares of order n.

When n is an odd integer, it is easy to see, by permuting a pair of rows or a pair of columns of a Latin square, that

$$\mathrm{RELS}(n)=\mathrm{ROLS}(n),$$

$$\mathrm{CELS}(n)=\mathrm{COLS}(n),$$

$$\mathrm{ELS}(n)=\mathrm{OLS}(n).$$

The following conjecture was made by Alon and Tarsi in 1986.

Conjecture 2. If n is even, then $\mathrm{ELS}(n)\neq\mathrm{OLS}(n)$.

We will show, using a formula in [7] that this conjecture is equivalent to the following.

Conjecture 3. If n is even, then $\mathrm{RELS}(n)\neq\mathrm{ROLS}(n)$.

It is obvious that Conjecture 3 is equivalent to

$$\mathrm{CELS}(n)\neq\mathrm{COLS}(n),$$

if n is even.

We now go on to the second author's conjectures.

Conjecture 4 (Rota, 1989). Let V be a vector space over an arbitrary infinite field. Suppose $B_1,B_2,\ldots,B_n$ are n sets of bases of V. Then for each i, there is a linear order of B_i, say $B_1=\{a_1,a_2,\ldots,a_n\}$; $B_2=\{b_1,b_2,\ldots,b_n\}$; ...; $B_n=\{c_1,c_2,\ldots,c_n\}$, such that $C_1=\{a_1,b_1,\ldots,c_1\}$; $C_2=\{a_2,b_2,\ldots,c_2\}$; ...; $C_n=\{a_n,b_n,\ldots,c_n\}$ are n sets of bases.

An analogous conjecture on bases of a matroid was made at the same time.

We shall see that Conjecture 4 is a consequence of the same author's conjecture concerning the nonvanishing of a certain element of the bracket algebra, which we proceed to state. (The definition of the bracket algebra will be recalled in next section.)

Conjecture 5 (Rota, 1989). Let Bracket $[L^+]$ be the bracket algebra of rank n on a set L^+ of positive letters. If n is even, the element of Bracket $[L^+]$

$$[ab\cdots c]^n$$

is non-zero whenever $a, b, \ldots, c$ are n distinct letters in L^+.

Conjecture 5 plays a central role in revealing the relations among Conjectures 1–5. We show that

$$\text{Conjecture 5} \overset{\text{(i)}}{\Leftrightarrow} \text{Conjecture 3} \overset{\text{(ii)}}{\Leftrightarrow} \text{Conjecture 2},$$

$$\text{Conjecture 5} \overset{\text{(iii)}}{\Rightarrow} \text{Conjecture 4 (for even } n\text{)},$$

$$\text{Conjecture 2} \overset{\text{(iv)}}{\Rightarrow} \text{Conjecture 1 (for even } n\text{)}.$$

Relation (i) is proved in Section 2, by a formula for straightening coefficients [7] in a bracket algebra. Relations (ii) and (iii) will be proved in Section 3, by the technique of umbral linear operators. Relations (iv) is proved in Section 4, as an immediate consequence of a theorem of Alon and Tarsi on list colorings of graphs.

In closing this section, we state a conjecture of Jeff Kahn which generalizes Conjectures 1 and 4.

Conjecture 6 (J. Kahn, 1991). Let V be an n dimensional vector space over an arbitrary infinite field. Associate to each cell in an $n\times n$ array a basis of V. Then it is always possible to choose one vector in each cell so that one gets bases in all rows and columns.

2. Bracket algebras and the straightening formula

We begin with reviewing the bracket algebra on a set of vectors. Let V be a vector space over a field K of characteristic zero. This assumption is made for ease of reading, although a characteristic free theory is also available, see [5, 6]. Fix a basis $e_1, e_2, \ldots, e_n$ of V. Given n vectors $x_1, x_2, \ldots, x_n$ in V, the *bracket* $[x_1\, x_2 \cdots x_n]$ is defined to be the determinant

$$[x_1\, x_2 \cdots x_n] = \det(x_{ij}),$$

where $x_i = \sum_j x_{ij} e_j$. It is clear that

$$[x_1 \cdots x_i \cdots x_j \cdots x_n] = -[x_1 \cdots x_j \cdots x_i \cdots x_n]. \tag{1}$$

Moreover for any $n+1$ vectors $x_1, x_2, \ldots, x_n, x_{n+1} \in V$, one can prove the following well-known syzygy for a fixed $1 \leqslant i \leqslant n$:

$$\sum_{\sigma \in S_{n+1}} (-1)^\sigma [x_{\sigma_1} x_{\sigma_2} \cdots x_{\sigma_i} y_1 y_2 \cdots y_{n-i}] [x_{\sigma_{i+1}} \cdots x_{\sigma_{n+1}} y_{n-i+1} \cdots y_{n-1}] = 0, \quad (2)$$

for any $n-1$ vectors $y_1, y_2, \ldots, y_{n-1} \in V$. Here $(-1)^\sigma$ is defined to be $(-1)^{i(\sigma)}$, where $i(\sigma)$ is the number of inversions in the permutation σ.

Let $L^- = \{\alpha, \beta, \gamma, \ldots\}$ be a set of vector variables in V. The *bracket algebra* Bracket$[L^-]$ is defined to be the commutative algebra generated by the brackets $[\alpha\beta \cdots \gamma]$, where $\alpha, \beta, \ldots, \gamma$ are n elements in L^-. The upper index $-$ on L^- is suggested by relation (1). We usually call elements in L^- *negative letters.*

Next, we define a different type of bracket algebra on a set $L^+ = \{a, b, c, \ldots, \}$, where the upper index $+$ is suggested by relation (4) below. Elements in L^+ will be called *positive letters.* Associate to L^+ a new set

$$[L^+] = \{[u] : u \text{ is a word on } L^+ \text{ of length } n\}.$$

The *bracket algebra* Bracket$[L^+]$ of rank n on the set L^+ of positive letters is the associative algebra generated by brackets $[u]$ in $[L^+]$ such that

(1) For any two brackets $[u]$ and $[v]$ in $[L^+]$,

$$[u][v] = (-1)^n [v][u]. \quad (3)$$

(2) For any two words u and v (of length n) on L^+ such that $u = v$ as monomials in the commutative algebra generated by L^+, then

$$[u] = [v]. \quad (4)$$

The above relation (in contrast to (1)) explains the use of the plus sign on L^+.

(3) For any $n+1$ letters $x_1, x_2, \ldots, x_{n+1} \in L^+$ (not necessarily distinct),

$$\sum_{\sigma \in S_{n+1}} [x_{\sigma_1} x_{\sigma_2} \cdots x_{\sigma_i} y_1 y_2 \cdots y_{n-i}] [x_{\sigma_{i+1}} \cdots x_{\sigma_{n+1}} y_{n-i+1} \cdots y_{n-1}] = 0, \quad (5)$$

for all choices of $n-1$ letters $y_1, y_2, \ldots, y_{n-1}$ in L^+.

The bracket algebras Bracket$[L^-]$ and Bracket$[L^+]$ are of fundamental importance in the study of invariants for tensors under the action of the general linear groups, see [6].

We next proceed to discuss the straightening formula. Suppose L^+ and L^- are linearly ordered sets. Given a rectangular array on L^- with n columns (often called as a *tableau* of rectangular shape)

$$T = \begin{matrix} \alpha & \beta & \cdots & \gamma \\ \alpha' & \beta' & \cdots & \gamma' \\ & & \vdots & \\ \alpha'' & \beta'' & \cdots & \gamma'' \end{matrix},$$

define

$$[T]=[\alpha\beta\cdots\gamma][\alpha'\beta'\cdots\gamma']\cdots[\alpha''\beta''\cdots\gamma'']. \tag{6}$$

Notice that if any letter occurs twice in a row of T, then $[T]=0$. A tableau T on a set L^- of negative letters is called *standard* if it has strictly increasing rows and weakly increasing columns.

Similarly, given a tableau T on L^+ of rectangular shape with n columns, we define $[T]$ as the product of brackets of its row words as in (6). However, the notion of a standard tableau is differently defined. A tableau T on a set L^+ of positive letters is now called *standard* if it has weakly increasing rows and strongly increasing columns.

The *content* of a tableau T on either L^- or L^+ is defined to be the multiset of the letters occurred in T.

Theorem 7 (straightening formula). *The set of all $[S]$, as S ranges over all standard tableaux of a given content on L^+ (resp. L^-) of the same rectangular shape with n columns, is a linear basis of the corresponding homogeneous subspace of the bracket algebra* Bracket$[L^+]$ *(resp.* Bracket$[L^-]$) *of rank n spanned by all $[T]$, as T ranges over all arbitrary tableaux of the same rectangular shape and of the same given content.*

Interested readers can refer [5] or [6] for a proof.

According to this theorem, given a tableau T on L^+ (resp. L^-) of a rectangular shape with n columns, one can write

$$[T]=\sum_S a_{TS}[S], \quad a_{TS}\in K,$$

where S ranges over standard tableaux on L^+ (resp. L^-) of the same shape and of the same content as T. The coefficients a_{TS} involved are called *straightening coefficients.*

We consider the algebra Bracket$[L^+]$. Given n letters $a<b<\cdots<c$ in L^+, let

$$T=\begin{matrix} a & b & \cdots & c \\ a & b & \cdots & c \\ & & \vdots & \\ a & b & \cdots & c \end{matrix}$$

be an $n\times n$ array on L^+. It is clear that T is not standard, since it has column repetitions. Therefore by the straightening formula we have

$$\begin{bmatrix} a & b & \cdots & c \\ a & b & \cdots & c \\ & & \vdots & \\ a & b & \cdots & c \end{bmatrix} = r \begin{bmatrix} a & a & \cdots & a \\ b & b & \cdots & b \\ & & \vdots & \\ c & c & \cdots & c \end{bmatrix} \tag{7}$$

for some rational number r, since the tableau on the right side is the only standard one with content $a^n b^n \cdots c^n$. The idea of this paper hinges on this mysterious coefficient r.

Its significance was first noticed by the second author. When n is odd, $r=0$ by (3). When n is even, Conjecture 5 is equivalent to

Conjecture 5′ (Rota, 1989). The coefficient r in (7) is non-zero.

We will prove later in this section and in the next section the identities

$$(n!)^n r = \mathrm{CELS}(n) - \mathrm{COLS}(n), \tag{8}$$

and

$$\pm(n!)^n r = \mathrm{ELS}(n) - \mathrm{OLS}(n). \tag{9}$$

These two identities establish the equivalence.

$$\text{Conjecture } 5' \Leftrightarrow \text{Conjecture } 3 \Leftrightarrow \text{Conjecture } 2.$$

Identity (8) will turn out to be a consequence of a formula in ref. 7. To arrive at this result, we require some definitions. Given a letter $a \in L$, let $a^{(i)}$ be the ith divided power. (Recall that a divided power $a^{(i)}$ is defined to be $a^i/i!$.) Let u be a word on L^+. The *stand* of u, denoted as stand(u), is the unique word on the set $\bigcup_{a\in L^+}\{a, a^{(2)}, a^{(3)}, \dots\}$ obtained from u by contracting consecutive identical letters into a divided power. For example,

$$\mathrm{stand}(abbcccbaa) = ab^{(2)}c^{(3)}ba^{(2)}.$$

Let T be a tableau of rectangular shape with n columns. Write the rows of T as $u_1, u_2, \dots, u_m$. These are words of length n on L^+. Define the *stand* of $[T]$, denoted as $[T]^s$, to be

$$[T]^s = [\mathrm{stand}\ u_1]\,[\mathrm{stand}\ u_2]\cdots[\mathrm{stand}\ u_m].$$

For example,

$$\begin{bmatrix} a & b & \cdots & c \\ a & b & \cdots & c \\ & & \vdots & \\ a & b & \cdots & c \end{bmatrix}^s = \begin{bmatrix} a & b & \cdots & c \\ a & b & \cdots & c \\ & & \vdots & \\ a & b & \cdots & c \end{bmatrix} = [ab\cdots c]^n$$

and

$$\begin{bmatrix} a & a & \cdots & a \\ b & b & \cdots & b \\ & & \vdots & \\ c & c & \cdots & c \end{bmatrix}^s = \frac{1}{(n!)^n}\begin{bmatrix} a & a & \cdots & a \\ b & b & \cdots & b \\ & & \vdots & \\ c & c & \cdots & c \end{bmatrix} = [a^{(n)}]\,[b^{(n)}]\cdots[c^{(n)}].$$

(In [5–7], $[T]^s$ is denoted as $[T]$.)

We recall the following result proved in [7].

Theorem 8. *Consider the bracket algebra* Bracket$[L^+]$. *Suppose*

$$[T]^s = \sum_{S \in \mathscr{A}} a_{TS} [S]^s$$

for some set $\mathscr{A}$ *of standard tableaux. Let* S *be that element in* $\mathscr{A}$ *such that its row sequence (the word obtained by writing the rows of* T *one after another from top to bottom) is the largest in the lexicographic order. Then*

$$a_{TS} = \sum_X \operatorname{sgn} X,$$

where the sum ranges over all tableaux X *(of the same shape as* S *and* T*) such that* X *is an interpolant from* T *to* S*, that is, a letter occurs in the ith row of* X *if and only if it occurs in the ith row of* T *and a letter occurs in the jth column of* X *if and only if it occurs in the jth column of* S*; and where* $\operatorname{sgn} X$ *is* $+1$ *or* -1 *depending on the total number of inversions occurred in columns of* X *been even or odd.*

Let us apply Theorem 8 to identity (7). It can be rewritten as

$$\begin{bmatrix} a & b & \cdots & c \\ a & b & \cdots & c \\ & & \vdots & \\ a & b & \cdots & c \end{bmatrix}^s = (n!)^n r \begin{bmatrix} a & a & \cdots & a \\ b & b & \cdots & b \\ & & \vdots & \\ c & c & \cdots & c \end{bmatrix}^s. \tag{10}$$

Let X be an interpolant from T to S, where T is the tableau on the left side of (10) and S is the one on the right. Clearly, X is a Latin square of order n. By definition, $\operatorname{sgn} X$ is exactly $(-1)^k$, where k is the number of column inversions of X. Therefore by Theorem 8,

$$(n!)^n r = \mathrm{CELS}(n) - \mathrm{COLS}(n). \tag{11}$$

This proves (8) and in turn implies the equivalence

Conjecture 5′ $\Leftrightarrow$ Conjecture 3.

3. Umbral linear operators

Umbral linear operators are important in invariant theory, in that they allow to write all invariants of a set of skew-symmetric tensors in terms of elements in Bracket$[L^+]$, see [6]. In this section, we use umbral operators to derive the implication

Conjecture 5′ $\Rightarrow$ Conjecture 4 (for even n),

and the equivalence

Conjecture 5′ $\Leftrightarrow$ Conjecture 2.

The definition of an umbral linear operator is ordinarily given in terms of polarization operators in a supersymmetric bracket algebra. We follow a more elementary approach and define these operators directly.

Let M be a finite multiset with elements in L^+. Suppose the size of M is a multiple of n. Write

$$M = a^i b^j \cdots c^k,$$

where $i+j+\cdots+k \equiv 0 \pmod{n}$. Let $\text{Bracket}[L^+]_M$ be the subspace of $\text{Bracket}[L^+]$ spanned by bracket monomials with content M. Iterating (5),

$$\text{Bracket}[L^+]_M = 0 \quad \text{unless } i, j, \ldots, k \leqslant n.$$

Given a monomial $m \in \text{Bracket}[L^+]_M$, write

$$m = m(a, a, \ldots, a; b, b, \ldots, b; \ldots; c, c, \ldots, c),$$

when the letters $a, b, \ldots, c$ occur $i, j, \ldots, k$ times respectively in the monomial m. Let V be a vector space of dimension n. Let

$$L^- = \{\alpha_1, \alpha_2, \ldots, \alpha_i; \beta_1, \beta_2, \ldots, \beta_j; \ldots; \gamma_1, \gamma_2, \ldots, \gamma_k\}$$

be a set of vector variables in V. Define the element $m^- \in \text{Bracket}[L^-]$ as

$$m^- = \sum_{\sigma, \tau, \cdots, \pi} (-1)^{\sigma+\tau+\cdots+\pi} m(\alpha_{\sigma_1}, \ldots, \alpha_{\sigma_i}; \beta_{\tau_1}, \ldots, \beta_{\tau_j}; \gamma_{\pi_1}, \ldots, \gamma_{\pi_k}).$$

In other words, we antisymmetrize the positive letters $a, b, \ldots, c$ by the negative letters $\alpha_1, \ldots, \alpha_i$; $\beta_1, \ldots, \beta_j$; $\ldots$; $\gamma_1, \ldots, \gamma_k$, respectively. The element m^- is well defined up to a sign. For example, if $n=2$, and if $m = [aab][bcc]$, then

$$m^- = \pm 4[\alpha_1\alpha_2\beta_1][\beta_2\gamma_1\gamma_2] \mp 4[\alpha_1\alpha_2\beta_2][\beta_1\gamma_1\gamma_2].$$

It can be proved that (see [6]) there exists a well-defined linear operator

$$\Phi : \text{Bracket}[L^+]_M \to \text{Bracket}[L^-],$$

called the *umbral linear operator*, such that for each monomial $m \in \text{Bracket}[L^+]_M$,

$$\Phi(m) = \pm m^-. \tag{12}$$

In fact, one can determine the sign in (12) explicitly. Fortunately, we do not have to worry about this sign in the present paper.

We now proceed to prove that Conjecture 5′ implies Conjecture 4 by using the umbral linear operator.

Suppose n is an even integer. Let the multiset M be

$$M = a^n b^n \cdots c^n$$

where $a, b, \ldots, c$ are n of letters L^+. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$; $\beta_1, \beta_2, \ldots, \beta_n$; $\ldots$; $\gamma_1, \gamma_2, \ldots, \gamma_n$ be n sets of vector variables in V. Applying the umbral operator Φ to each side of identity

(7), one obtains the identity

$$\sum_{\sigma,\tau,\cdots,\pi\in S_n}(-1)^{\sigma+\tau+\cdots+\pi}\begin{bmatrix}\alpha_{\sigma_1} & \beta_{\tau_1} & \cdots & \gamma_{\pi_1}\\ \alpha_{\sigma_2} & \beta_{\tau_2} & \cdots & \gamma_{\pi_2}\\ & & \vdots & \\ \alpha_{\sigma_n} & \beta_{\tau_n} & \cdots & \gamma_{\pi_n}\end{bmatrix}=\pm(n!)^n r\begin{bmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_n\\ \beta_1 & \beta_2 & \cdots & \beta_n\\ & & \vdots & \\ \gamma_1 & \gamma_2 & \cdots & \gamma_n\end{bmatrix}. \tag{13}$$

Now evaluate the n sets of vector variables as n sets of bases of V. The tableau on the right side of (13) does not vanish, since the entries on each row are a basis, and the bracket of a basis is a non-zero number. Thus if $r\neq 0$ then the left side of (13) does not vanish either. Therefore at least one term in the sum on the left side of (13) is non-zero; say such a term is

$$\begin{bmatrix}\alpha_1 & \beta_1 & \cdots & \gamma_1\\ \alpha_2 & \beta_2 & \cdots & \gamma_2\\ & & \vdots & \\ \alpha_n & \beta_n & \cdots & \gamma_n\end{bmatrix}=[\alpha_1\beta_1\cdots\gamma_1][\alpha_2\beta_2\cdots\gamma_2]\cdots[\alpha_n\beta_n\cdots\gamma_n]\neq 0.$$

This implies that for $i=1,2,\ldots,n$, $\{\alpha_i,\beta_i,\ldots,\gamma_i\}$ is a basis of V. We have therefore proved that Conjecture 5′ implies Conjecture 4.

Next, we prove that Conjecture 5′ is equivalent to Conjecture 2. In the same notation as above, evaluate the n sets of vector variables such that

$$\alpha_i=\beta_i=\cdots=\gamma_i,\quad i=1,2,\ldots,n,$$

and that

$$[\alpha_1\alpha_2\cdots\alpha_n]=1. \tag{14}$$

Under these assumptions, identity (13) simplifies to

$$\sum_{\sigma,\tau,\cdots,\pi\in S_n}(-1)^{\sigma+\tau+\cdots+\pi}\begin{bmatrix}\alpha_{\sigma_1} & \alpha_{\tau_1} & \cdots & \alpha_{\pi_1}\\ \alpha_{\sigma_2} & \alpha_{\tau_2} & \cdots & \alpha_{\pi_2}\\ & & \vdots & \\ \alpha_{\sigma_n} & \alpha_{\tau_n} & \cdots & \alpha_{\pi_n}\end{bmatrix}=\pm(n!)^n r\begin{bmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_n\\ \alpha_1 & \alpha_2 & \cdots & \alpha_n\\ & & \vdots & \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n\end{bmatrix} \tag{15}$$

Note that a term on the left side of (15) is non-zero if and only if its tableau is a Latin square. Moreover, by (14) such a non-zero term

$$(-1)^{\sigma+\tau+\cdots+\pi}\begin{bmatrix}\alpha_{\sigma_1} & \alpha_{\tau_1} & \cdots & \alpha_{\pi_1}\\ \alpha_{\sigma_2} & \alpha_{\tau_2} & \cdots & \alpha_{\pi_2}\\ & & \vdots & \\ \alpha_{\sigma_n} & \alpha_{\tau_n} & \cdots & \alpha_{\pi_n}\end{bmatrix}$$

has a sign $+1$ or -1 depending on whether it is an even or odd Latin square. Hence identity (15) gives

$$\mathrm{ELS}(n)-\mathrm{OLS}(n)=\pm(n!)^n r.$$

We infer that $r\neq 0$ if and only if $\mathrm{ELS}(n)\neq\mathrm{OLS}(n)$.

Remark. By polarization techniques, it can be shown that when n is even,

$$\mathrm{ELS}(n)-\mathrm{OLS}(n)=(-1)^{\frac{n(n-1)}{2}}(n!)^n r.$$

Taking transposes, we obtain

$$\mathrm{CELS}(n)-\mathrm{COLS}(n)=\mathrm{RELS}(n)-\mathrm{ROLS}(n).$$

Combining with identity (11), we conclude

$$\mathrm{ELS}(n)-\mathrm{OLS}(n)=\mathrm{CELS}(n)-\mathrm{COLS}(n)=\mathrm{RELS}(n)-\mathrm{ROLS}(n)$$

when $n=4k$,

$$\mathrm{OLS}(n)-\mathrm{ELS}(n)=\mathrm{CELS}(n)-\mathrm{COLS}(n)=\mathrm{RELS}(n)-\mathrm{ROLS}(n)$$

when $n=4k+2$.

Denote by $\mathrm{LS}(e,e,n)$ the number of row even and column even Latin squares of size n, $\mathrm{LS}(e,o,n)$ the number of row even and column odd Latin squares of size n, etc. Then a simple computation leads to the following identity:

$$\mathrm{LS}(o,e,n)=\mathrm{LS}(e,o,n)=\mathrm{LS}(o,o,n),\quad \text{when } n=4k,$$

$$\mathrm{LS}(o,e,n)=\mathrm{LS}(e,o,n)=\mathrm{LS}(e,e,n),\quad \text{when } n=4k+2.$$

4. List colorings of graphs

In this section it will be shown that the Alon–Tarsi Conjecture implies the Dinitz Conjecture for even n. We follow largely [8].

Given a graph G on vertex set $[n]$ and n arbitrary sets $S=\{S_i : i\in[n]\}$, an *S-legal vertex coloring* of G is a function $\sigma:[n]\to\Pi_i S_i$ such that $i\to S_i$ is a proper coloring of G in the ordinary sense. A digraph D is called *Eulerian* if the in-and-out-degrees $d^-(v)$ and $d^+(v)$ coincide for each vertex v, and *even* (*odd*) if it has an even (odd) number of oriented edges. We write $\mathrm{EE}(D)$ and $\mathrm{OE}(D)$ for the number of even and odd Eulerian subdigraphs of D. The main result of [1] states the following.

Theorem 9 (Alon–Tarsi). *If G has an orientation D with out-degree sequence $\boldsymbol{d}(D)=\{d^+(1),d^+(2),\dots,d^+(n)\}$ such that*

$$d^+(i)<|S_i| \quad \textit{for all } i, \textit{ and} \tag{16}$$

$$\mathrm{EE}(D)\neq \mathrm{OE}(D), \tag{17}$$

then G has an S-legal vertex coloring.

We proceed to deduce from this theorem that the Alon–Tarsi conjecture implies the Dinitz Conjecture when n is even. We work with the graph G on the vertex set $V=\{(i,j):1\leqslant i,j\leqslant n\}$, where two vertices are adjacent if they agree in one coordinate. Suppose $|S_{ij}|=n$ for all i,j. Each Latin square $L: V\to[n]$ gives an orientation D_L of G with all out-degrees $n-1$ according to

$$(i,j)\to(i,j') \quad \text{if } L(i,j)<L(i,j'),$$

$$(i,j)\to(i',j) \quad \text{if } L(i,j)>L(i',j).$$

Now we take $D=D_{L_0}$ in the above theorem, where $L_0(i,j)\equiv i+j-1 \pmod n$. Then condition (16) is satisfied. As for condition (17) in the theorem, one notices that

$$\boldsymbol{d}(D')=\boldsymbol{d}(D) \Leftrightarrow D\backslash D' \text{ is an Eulerian subdigraph of } D,$$

where $D'\backslash D$ refers to the set of oriented edges on which D and D' disagree. Hence

$$\mathrm{EE}(D)-\mathrm{OE}(D)=\sum_{D':\boldsymbol{d}(D')=\boldsymbol{d}(D)}(-1)^{|D'\backslash D|}. \tag{18}$$

Now let D_0 be the orientation such that

$$(i,j)\to(i,j') \quad \text{if } j<j',$$

$$(i,j)\to(i',j) \quad \text{if } i>i'.$$

Then it is not hard to check that for all Latin squares L,

(i) $|D_L\backslash D_{L_0}|\equiv|D_L\backslash D_0|+|D_0\backslash D_{L_0}| \pmod 2$,

(ii) $|D_L\backslash D_0|=$ the number of inversions in L,

(iii) $|D_{L_0}\backslash D_0|$ is even.

Therefore $|D_L\backslash D_{L_0}|\equiv$ the number of inversions in $L \pmod 2$.

To complete the argument, one sees that the net contribution of all other orientations having all out degrees $n-1$ to (18) is zero. Therefore by (18),

$$\mathrm{EE}(D_{L_0})-\mathrm{OE}(D_{L_0})=\mathrm{ELS}(n)-\mathrm{OLS}(n).$$

So by Theorem 9, the Alon–Tarsi Conjecture implies the Dinitz Conjecture for even n.

In closing, we recall a well-known conjecture on list colorings of graphs, which can be regarded as a generalization of the Dinitz Conjecture. The history of this conjecture is given in [2].

Let G be a multigraph with edge set E. Denote by $\chi'(G)$ the edge coloring number of G. The *edge-list-coloring number* of G, denoted $\chi_l'(G)$, is the least t such that if each edge

$e \in E$ is assigned a list $S(e)$ of t 'legal' colors, then there is a coloring of the edges of E by $\bigcup\{S(e): e \in E\}$ which is a proper coloring in the usual sense, and which assigns each e a color from $S(e)$.

Conjecture 10 (List coloring conjecture). $\chi_l'(G) = \chi'(G)$ for every multigraph G.

Obviously, the Dinitz Conjecture is the special case of the above conjecture when G is the bipartite graph $K_{n,n}$.

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