Around two theorems and a lemma by Lucio Russo

Itai Benjamini and Gil Kalai

1 Introduction

We did not meet Lucio Russo in person but his mathematical work greatly influenced our own and his wide horizons and interests in physics, mathematics, philosophy, and history greatly inspired us. We describe here two directions of study following early work of Russo. The first section follow the famous Russo-Seymour-Welsh theorem regarding critical planar percolation. The second section follows the basic "Russo's lemma" and deep "Russo's 0-1 law". In each direction we present one central conjecture.

2 Planar percolation

Consider Bernoulli percolation on a planar grid. Russo [23] and Seymour and Welsh [25] proved the *RSW theorem* comparing the probability of crossing a rectangle to that of crossing a square. In particular implying that:

the probability critical percolation crosses a long rectangle is bounded away from zero and is depending only on the aspect ratio.

This fundamental fact is used and extended to a variety of models, involving rather clever proofs. Vincent Tassion recently gave a couple of proofs for the RSW fact under various sets of weaker assumptions, key to solving several known problems [29]. Personally let us note that the RSW lemma was essential in controlling the influence of a fixed edge on the crossing event, allowing us to establish, jointly with Oded Schramm, noise sensitivity of critical percolation, see [2, 15].

What about a RSW type result for a more general planar graphs going beyond Euclidean lattices and tessellations?

In what follows we would like to suggest a conjectural extension of RSW theorem to general planar triangulations. The motivation comes from conformal uniformization, see [1].

2.1 A generalized RSW conjecture.

Tile the unit square with (possibly infinity number) of squares of varying sizes so that at most three squares meet at corners. Color each square black or white with equal probability independently.

Conjecture 2.1. *Show that there is a universal $c > 0$, so that the probability of a black left right crossing is bigger than c.*

If true, the same should hold for a tiling, or a packing of a triangulation, with a set of shapes that are of bounded Hausdorff distance to circles. At the moment we don't have a proof of the conjecture even when the squares are colored black with probability $2/3$.

Behind the conjecture is a coarse version of conformal invariance. That is, the crossing probability is balanced if the tiles are of uniformly bounded distance to circles (rotation invariance), and the squares can be of different sizes, (dilation invariance).

How does the *influence* of a square in the tiling on the crossing probability at $p = 1/2$ and it's area related?

If the answer to conjecture 2.1 is affirmative, this will imply the following:

Let G be the 1-skeleton of a bounded degree triangulation of an open disk. Assume G is transient for the simple random walk, then 1/2-Bernoulli site percolation on G admits infinitely many infinite cluster a.s. We don't know it even for any fixed $p > 1/2$. By [3] such triangulations result in a square tilings as in the conjecture. the proof in [3] is an analogous of the RSW phenomena for simple random walk on the triangulation. We speculate 1/2-Bernoulli site percolation on G admits infinitely many infinite cluster a.s. *iff* G is transient.

Establishing a *high dimensional* version of the RSW lemma is a well known very important open problem. Either at $p = 1/2$ for two dimensional plaquettes in four dimension using duality, or in three dimension place independently 2 cells in the $3D$ square grid. Look at the critical p for full infinite surface and prove RSW for plaquetts in cubes. That is, if the probability of no open path from top to bottom in a $n \times n \times n$ box with probability at least $1/2$, then no open path from top to bottom in a cube $2n \times 2n \times n$, with probability bounded away from 0 independently of n.

A comment on large graphs and percolation. In the category of planar graphs, in view of the (discrete) conformal uniformization, transience (equivalently *conformally hyperbolic*) is a natural notion of largeness. In the context of Cayley graphs *nonamenability* serves as a notion of large Cayley graphs. Thus the still open conjecture ([4]) that there is a non empty interval of p's so that $p-$ Bernoulli percolation admits infinitely many infinite clusters *iff* the group is nonamenable, shares some flavor with conjecture 2.1. As both suggest that a graph is large provided there is a phase with infinitely many infinite clusters.

3 Isoperimetric inequalities and Russo's 0-1 law

We endow the discrete cube $\Omega_n = \{-1,1\}^n$ with the probability product measure μ_p where the probability for each bit to be 1 is p. A Boolean function f is a function from Ω_n to $\{-1,1\}$, and f is monotone if changing the value of a variable from -1 to 1 does not change the value of f from 1 to -1. The influence of the kth variable on f denoted by $I_k^p(f)$ is the probability that changing the kth variable will change the value of f. The total influence is $I^p(f) = \sum_{k=1}^n I_k^p(f)$. We denote $\mu_p(f) = \mu_p\{x : f(x) = 1\}$, and write $var_p(f) = \mu_p(f)(1-\mu_p(f))$. (If $p = 1/2$ we omit the superscript/subscript p.)

A basic result in extremal and probabilistic combinatorics going back to Harper (and others) is the isoperimetric inequality. For the measure μ_p the isoperimetric relation takes the form:

Theorem 3.1.

$$pI^p(f) \geq \mu_p(f)\log_p(\mu_p(f)).$$

If f is monotone then $\mu_p(f)$ is a monotone function of p. Fixing a small $\epsilon > 0$ the threshols interval of f is the interval $[p,q]$ where $\mu_p(f) = \epsilon$, $\mu_{1-p}(f) = 1-\epsilon$. A fundamental lemma by Russo [24] and Margulis [21] asserts that for a monotone Boolean function f,

$$d\mu_p(f)/dp = I^p(f).$$

The deep Russo's 0-1 law [24] asserts informally that the threshold interval of a Boolean function is of size $o(1)$ if all variables have $o(1)$-influence. In view of the Russo-Margulis lemma understanding the total influence is crucial for understanding the threshold window of a Boolean function. Sharp form of Russo 0-1 theorem and various related results were proved in the last two decades and Fourier methods played an important role in these developments. We mention especially the paper by Kahn, Kalai and Linial [18] and subsequent papers [8, 27, 14, 12], and books [15, 22]. To a large extent, this study is centered around the following problem.

Problem: Understand the structure of Boolean functions of n variables for which

$$I^p(f) \leq K\frac{1}{p}\mu_p(f)\log_p(1/\mu_p(f)).$$

We will quickly describe some main avenues of research and central results regarding this problem.

1. For the case where k is bounded, both p and $\mu_p(f)$ are bounded away from zero and one (or even when $\log(1/p)/\log n \to 0$) and K is bounded, Friedgut (1998) [12] proved that such functions are approximately "juntas," namely, they are determined (with high probability) by their values on a fixed bounded set of variables. This result can be seen as a sharp form of Russo's 0-1 law and it has a wide range of applications.

2. For the case where K is bounded, $\mu_p(f)$ is bounded away from zero and one, but $\log p/\log n$ is bounded away from zero, there important theorems by Friedgut (1999) [13], Bourgain (1999) [7] (see below), and Hatami (2010) [17]. Hatami's work is based on a mysterious while important notion of pseudojuntas.

3. The case where K is bounded, and $\mu_p(f)$ is small is wide open and very important.

4. Cases where $K = 1 + \epsilon$ are of different nature and also of importance. See, e.g. [10, 11].

5. There are few results regarding the case that K is unbounded and especially when K grows quicker than $\log n$. (One such result is by Bourgain and Kalai for functions with various forms of symmetry [9].) This is of great interest already when both p and $\mu_p(f)$ are bounded away from 0 and 1.

We will mention now a theorem of Bourgain and a far-reaching related conjecture.

Theorem 3.2 (Bourgain [7])**.** *There exists $\epsilon > o$ with the following property. For every C there is $K(C)$, such that if $I^p(f) < pC$ then there exists a subset R of variables $|R| \leq K(C)$ such that*

$$\mu_p(x : f(x) = 1 | x_i = 1, i \in S) > (1 + \epsilon)\mu_p(f).$$

Conjecture 3.3 (Kahn and Kalai Conjecture 6.1 (a) from [19])**.** *There exists $\epsilon > 0$ with the following property. For every C there is $K(C)$, such that if $I^p(f) < pC\mu_p(f)\log(1/\mu_p(f))$ then there exist a subset R of variables $|R| \leq K(C)\log(1/\mu_p(f))$ such that*

$$\mu_p(x : f(x) = 1 | x_i = 1, i \in S) > (1 + \epsilon)\mu_p(f).$$

Several attemts for stronger conjectures (such as Conjectures 6.1(b), 6.1 (c) from [19]) turned out to be incorrect. We conclude with another approach for understanding Boolean functions with small influence. The first step is the important Fourier-Walsh expansion.

Every Boolean function f can be written as a square free polynomial $f = \sum \hat{f}(S)x_S$, where $x_S = \prod\{x_i : i \in S\}$. (The coefficients $\hat{f}(S)$ are called the Fourier coefficients of f.) It is easy to verify that $\sum \hat{f}^2(S) = 1$ and that $\sum \hat{f}^2(S)|S| = I(S)$. It follows that

Proposition 3.4. *For every $\epsilon > 0$ a Boolean function f can be $\epsilon \cdot var(f)$-approximated by the sign of a degree-d polynomial where $d = \frac{1}{\epsilon}I(f)$.*

However, we note that Boolean functions described as signs of low degree polynomials may have large total influence. Our next step is to consider representation of Boolean functions via Boolean circuits. Circuits allow to build complicated Boolean functions from simple ones and they have crucial importance in computational complexity. Starting with n variables $x_1, x_2, \ldots, x_n$, a *literal* is a variable x_i or its negation $-x_i$. Every Boolean function can be written as a formula in conjunctive normal form, namely as AND of ORs of literals. A *circuit* of depth d is defined inductively as follows. A circuit of depth zero is a literal. A circuit of depth one consists of an OR or AND *gate* applied to a set of literals, a circuit of depth k consists of an OR or AND gate applied to the outputs of circuits of depth $k-1$. (We can assume that gates in the odd levels are all OR gates and that the gates of the even levels are all AND gates.) The size of a circuit is the number of gates. The famous **NP** $\neq$ **P**-conjecture (in a slightly stronger form) asserts that the Boolean function described by the graph property of containing a Hamiltonial cycle, cannot be described by a polynomial-size circuit.

A theorem by Boppana [6] (the monotone case) and Hastad [16] (the general case) asserts that if f is described bt a depth d size M Boolean circuit then $I(f) \leq C(\log M)^{d-1}$. We conjecture that functions with low influence can be approximated by low-depth small size circuits.

Conjecture 3.5 (Benjamini, Kalai, and Schramm [2], 1999 (Slightly extended))**.** *For some absolute constant C the following holds. For every $\epsilon > 0$ a Boolean function f can be $\epsilon \cdot var(f)$-approximated by a circuit of depth d of size M where*

$$(\log M)^{Cd}\ var(f) \leq I(f).$$

4 Conclusion

Congratulations, Lucio for your remarkable career and contributions and best wishes for the future. It is time for us to meet!

References

[1] I. Benjamini, Percolation and coarse conformal uniformization, http://arxiv.org/abs/1510.05196

[2] I. Benjamini, G. Kalai and O. Schramm, Noise sensitivity of Boolean functions and applications to percolation. IHES Publ. Math. 90 (1999) 5–43.

[3] I. Benjamini and O. Schramm, Random walks and harmonic functions on infinite planar graphs using square tilings. Ann. Probab. 24 (1996) 1219–1238.

[4] I. Benjamini and O. Schramm, Percolation beyond Zd, many questions and a few answers. Electron. Comm. Probab. 1 (1996) 71-82.

[5] B. Bollobás and A. Thomason, Threshold functions, *Combinatorica* 7 (1987), 35–38.

[6] R. Boppana, Threshold functions and bounded depth monotone circuits, *Proceedings of 16th Annual ACM Symposium on Theory of Computing* (1984), 475–479.

[7] J. Bourgain, On sharp thresholds of monotone properties, *J. Amer. Math. Soc.* 12 (1999), 1051–1054.

[8] J. Bourgain, J. Kahn, G. Kalai, Y. Katznelson and N. Linial, The influence of variables in product spaces, *Isr. J. Math.* 77 (1992), 55–64.

[9] J. Bourgain and G. Kalai, Influences of variables and threshold intervals under group symmetries, *Geom. Funct. Anal.* 7 (1997), 438–461.

[10] D. Ellis, Almost isoperimetric subsets of the discrete cube, arXive:1310.8179.

[11] D. Ellis and N. Lifshitz, work in progress.

[12] E. Friedgut, Boolean functions with low average sensitivity depend on few coordinates, *Combinatorica* 18 (1998), 27–35.

[13] E. Friedgut, Sharp thresholds of graphs properties, and the k-sat problem, *Jour. Amer. Math. Soc.* 12 (1999), 1017–1054.

[14] E. Friedgut and G. Kalai, Every monotone graph property has a sharp threshold, *Proc. Amer. Math. Soc.* **124** (1996), 2993–3002.

[15] C. Garban, Christophe and J. Steif, Noise Sensitivity of Boolean Functions and Percolation, Cambridge university press 2014.

[16] J. Håstad, Almost optimal lower bounds for small depth circuits, in *Randomness and Computation*, 5, ed. S. Micali, (1989), 143–170.

[17] H. Hatami, A structure theorem for Boolean functions with small total influences, *Annals of Mathematics*, 176 (2012), 509–533.

[18] J. Kahn, G. Kalai, and N. Linial, The influence of variables on Boolean functions, in *Proc. 29-th Annual Symposium on Foundations of Computer Science* (1988), 68–80.

[19] J. Kahn and G. Kalai, Thresholds and expectation thresholds, *Combin. Probab. Comput.* 16 (2007), 495–502.

[20] H. Kesten, The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$, *Comm. Math. Phys.* 74 (1980), 41–59.

[21] G. Margulis. Probabilistic characteristics of graphs with large connectivity (in Russian), *Probl. Pered. Inform.* 10 (1974), 101–108.

[22] R. O'Donnell, *Analysis of Boolean Functions,* Cambridge University Press (2014).

[23] L. Russo, A note on percolation, *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* 43 (1978), 39–48.

[24] L. Russo, An approximate zero-one law, *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* 61 (1982), 129–139.

[25] P. D. Seymour and D. J. A. Welsh, Percolation probabilities on the square lattice, Advances in Graph Theory (B. Bollobás, ed.), Annals of Discrete Mathematics 3, North-Holland, Amsterdam, 1978, pp. 227–245.

[26] S. Smirnov, Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits. C. R. Acad. Sci. Paris Ser. I Math. 333 (2001) 239–244.

[27] M. Talagrand, On Russo's approximate zero-one law, *Annals of Probability*, 22 (1994), 1576–1587.

[28] M. Talagrand, Are many small sets explicitely small, *STOC* 2009.

[29] V. Tassion, Crossing probabilities for Voronoi percolation, Ann. of Probab. 44 (2016) 3385-3398.