Problems for Imre Bárány's Birthday

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I will give several problems in the interface between combinatorics and geometry and mainly around Helly's theorem and Tverberg's theorem. This is one of the areas on which Imre Bárány had immense impact. My lecture in the conference will focus on some of these problems.

1 Around Tverberg's theorem

Tverberg's Theorem states the following: Let $x_1, x_2, \ldots, x_m$ be points in $\mathbb{R}^d$ with $m \geq (r-1)(d+1)+1$. Then there is a partition $S_1, S_2, \ldots, S_r$ of $\{1, 2, \ldots, m\}$ such that $\cap_{j=1}^{r} conv(x_i : i \in S_j) \neq \emptyset$. This was a conjecture by Birch who also proved the planar case. The bound of $(r-1)(d+1)+1$ in the theorem is sharp as can easily be seen from configuration of points in sufficiently general position. The case $r = 2$ is Radon's theorem.

1.1 Prescribing the sizes of parts in Tverberg's theorem and the number of Tverberg's partitions

Problem 1 *Suppose that $a_1, a_2, \ldots, a_r$ is a partition of $m = (r-1)(d+1)+1$ such that $1 \leq a_i \leq d+1$ for every i. Is there a configuration of m points in $\mathbb{R}^d$ of which all of Tverberg's partitions are of type $(a_1, a_2, \ldots, a_r)$?*

This problem was raised by Micha A. Perles many years ago and a positive answer was recently given by Moshe White.

Conjecture 2 (Sierksma Conjecture) *The number of Tverberg's r-partitions of a set of $(r-1)(d+1)+1$ points in $\mathbb{R}^d$ is at least $((r-1)!)^d$.*

White's examples provide a rich family of examples for cases of equality in Sierksma's conjecture. An even more general family of constructions for equality cases, based on stairway convexity, was given by Boris Bukh, Po-Shen Lo, and Gabriel Nivasch.

1.2 Topological Tverberg

Conjecture 3 (Topological Tverberg) *Let f be a continuous function from the m-dimensional simplex σ^m to $\mathbb{R}^d$. If $m \geq (d+1)(r-1)$ then there are r pairwise disjoint faces of σ^m whose images have a point in common.*

If f is a linear function this conjecture reduces to Tverberg's theorem. The case $r = 2$ was proved by Bajmoczy and Bárány using the Borsuk-Ulam theorem. In this case you can replace the simplex by any other polytope of the same dimension. The case where r is a prime number was proved in a seminal 1978 paper of Bárány, Shlosman and Szucs. The prime power case was proved by Özaydin. For the prime power case, the proofs are quite difficult and are based on computations of certain characteristic classes.

In 2015 the topological Tverberg conjecture was disproved. This involves some early result on vanishing of topological obstructions by Özaydin, a theory developed by Mabillard and Wagner extending Whitney's trick to k-fold intersections, and a fruitful reduction by Gromov and by Blagojević, Frick and Ziegler.

Conjecture 4 *Let f be a linear function from an m-dimensional polytope P to $\mathbb{R}^d$. If $m \geq (d+1)(r-1)$ then there are r pairwise disjoint faces of P whose images have a point in common.*

Problem 5 *Does the conclusion of the topological Tverberg conjecture holds if the images of faces under f form a "good cover?" (Namely, all those images and all non empty intersections are contractible.)?*

1.3 Colorful Tverberg

Let $C_1, \cdots, C_{d+1}$ be disjoint subsets of $\mathbb{R}^d$, called colors, each of cardinality at least t. A $(d+1)$-subset S of $\bigcup_{i=1}^{d+1} C_i$ is said to be multicolored if $S \cap C_i \neq \emptyset$ for $i = 1, \cdots, d+1$. Let r be an integer, and let $T(r, d)$ denote the smallest value t such that for every collection of colors $C_1, \cdots, C_{d+1}$ of size at least t there exist r disjoint multicolored sets $S_1, \cdots, S_r$ such that $\bigcap_{i=1}^{r} \operatorname{conv}(S_i) \neq \emptyset$.

A seminal theorem of Zivaljevic and Vrecica asserts that $T(r, d) \leq 4r - 1$ for all r, and $T(r, d) \leq 2r - 1$ if r is a prime. The only known proofs for this theorem rely on topological arguments.

Conjecture 6 (Bárány-Larman's colorful Tverberg conjecture)

$$T(r, d) = r.$$

The case where $r + 1$ is a prime was proved by Blagojevic, Matschke, and Ziegler.

Colorful Caratheodory and the Rota basis conjecture

Consider $d + 1$ sets $A_1, A_2, \cdots, A_{d+1}$ of points in $\mathbb{R}^d$. Assume that each $|A_i| = d + 1$ and that the interior of $conv(A_i)$ contains the origin.

Problem 7 (D. H. J. Polymath) *Can we find a partition of all points into $d+1$ rainbow parts such that the interior of the convex hulls of the parts have a point in common. (A rainbow set is a set containing one element from each A_i.)*

This question was raised in Chow's polymath12 dealing with Rota's basis conjecture. Note that Bárány's famous colorful Caratheodory asserts that there is a rainbow set whose convex hull contains the origin. (I don't know what is the maximum guaranteed number of disjoint rainbow sets with this property.) Without the words "the interiors of" this is a special case of the colorful Tverberg conjecture. A positive answer would be a strong variant of Reay's conjecture (below) on the dimension of Tverberg points, and also a strong form of (a somewhat special case) of Rota's basis conjecture.

1.4 Eckhoff's partition conjecture

Let X be a set endowed with an abstract closure operation $X \to cl(X)$. The only requirements of the closure operation are:

(1) $cl(cl(X)) = cl(X)$ and

(2) $A \subset B$ implies $cl(A) \subset cl(B)$.

Define $t_r(X)$ to be the largest size of a (multi)set in X which cannot be partitioned into r parts whose closures have a point in common.

Conjecture 8 (Eckhoff's Partition Conjecture:) *For every closure operation*

$$t_r \leq t_2 \cdot (r-1).$$

If X is the set of subsets of $\mathbb{R}^d$ and $cl(A)$ is the convex hull operation then Radon's theorem asserts that $t_2(X) = d+1$ and Eckhoff's partition conjecture would imply Tverberg's theorem. In 2010 Eckhoff's partition conjecture was refuted by Boris Bukh. Bukh's beautiful paper contains several important ideas and further results. I will mention one ingredient: Let me take for granted the nerve construction for moving from a family of n convex sets to a simplicial complex with n vertices recording their empty and non empty intersections. Bukh studied simplicial complexes whose vertex sets correspond to the power set of a set of size n: Starting with n points in $\mathbb{R}^d$ or some abstract convexity space consider the nerve of convex hulls of all subsets of these points!

1.5 Dimensions of Tverberg's point

A conjecture of Reay

For a set A, denote by $T_r(A)$ those points in $\mathbb{R}^d$ which belong to the convex hull of r pairwise disjoint subsets of A. We call these points Tverberg points of order r.

Conjecture 9 (Reay) *If A is a set of $(d+1)(r-1)+1+k$ points in general position in $\mathbb{R}^d$ then*

$$dim T_t(A) \geq k$$

.

In particular, Reay conjecture asserts that a set of $(d+1)r$ points in general position in $\mathbb{R}^d$ can be partitioned into r sets of size $d+1$ so that the simplices described by these sets have an interior common point.

The cascade conjecture

Conjecture 10 *For every $A \subset \mathbb{R}^d$,*

$$\sum_{r=1}^{|A|} \dim T_r(A) \geq 0.$$

(Note that $\dim \emptyset = -1$.) The conjecture was proved for $d \leq 2$ by Akiva Kadari (unpublished M. Sc thesis in Hebrew).

A special case

A special case of the cascade Conjecture asserts that given $2d + 2$ points in $\mathbb{R}^d$ then you can either partition them into two simplices whose interior intersects, or you can find a Tverberg partition into 3 parts. A reformulation based on positive hulls is:

Given $2d$ non zero vectors in $\mathbb{R}^d$ so that the origin is a vertex of the cone spanned by them then either:

- You can divide the points into two sets A and B so that the cones spanned by them have a d-dimensional intersection, or
- You can divide them into three sets A B and C so that the cones spanned by them have a non-trivial intersection.

Another interesting reformulation is obtained when we dualize using the Gale transform, and this have led to the problem we consider next.

A question about directed graphs that can be described as the union of two trees

A very special class of configurations arise from graphs. Start from a directed graph on n vertices and $2n-2$ edges and associate to each directed edge $\{i, j\}$ the vector $e_j - e_i$. This has led to the problem we discuss next.

Problem 11 *Let G be a directed graph with n vertices and $2n - 2$ edges. When can you divide your set of edges into two trees T_1 and T_2 (so far we disregard the orientation of edges,) so that when you reverse the directions of all edges in T_2 you get a strongly connected digraph.*

I conjectured that if G can be written as the union of two trees, the only additional obstruction is that there is a cut consisting only of two edges in reversed directions. Maria Chudnovsky and Paul Seymour found an additional necessary condition: There is no induced cycle $c_1 - ... - c_{2k} - c_1$ in G, s.t. each c_i is cubic, the edges of the cycle alternate in direction, and none of $c_1, .., c_{2k}$ are sources or sinks of G.

1.6 Another conjecture by Reay

Problem 12 *What is the smallest integer $R(d,r)$ such that If $x_1, x_2, \ldots, x_m$ be points in $\mathbb{R}^d$ with $m \geq R(d,r)$, then there is a partition $S_1, S_2, \ldots, S_r$ of $\{1,2,\ldots,m\}$ such that $conv(x_i : i \in S_j) \cap conv(x_i : i \in S_k) \neq \emptyset$, for every $1 \leq j < k \leq r$.*

Reay conjectured that you cannot improve the value given by Tverberg's theorem, namely that

Conjecture 13 (Reay) $R(d,r) = (r-1)(d+1)+1$.

Micha A. Perles conjectures that Reay's conjecture is false even for $r = 3$ for large dimensions, but with Moria Sigron he proved the strongest positive results in the direction of Reay's conjecture.

1.7 Another old problem

Problem 14 *How many points $T(d;s,t)$ in $\mathbb{R}^d$ guarantee that they can be divided into two parts so that every union of s convex sets containing the first part has a non empty intersection with every union of t convex sets containing the second part.*

I would like to explain why $R(d;s,t)$ is finite. This is a fairly general Ramsey-type argument and it gives us an opportunity to mention a few recent important results. The argument has two parts:

1) Prove that $T(d;s,t)$ is finite (with good estimates) when the points are in cyclic position.

2) Use the fact that for every d and m there is $f(d,m)$ so that among every n points in general position in $\mathbb{R}^d$, $n > f(d,n)$ one can find m points in cyclic position.

The finiteness follows (with horrible bounds) from these two ingredients by standard Ramsey-type results.

Recently a fairly good understanding of $f(d,n)$ was achieved in a series of beautiful papers,

Theorem 1

$$f(d,n) = twr_d(\theta(n).)$$

Here, twr_d is the d-fold tower function. The lower bound is by Suk (improving earlier bounds by Conlon, Fox, Pach, Sudakov and Suk) and the upper bounds are by Bárány, Matousek, and Por.

2 Helly and fractional Helly

2.1 A conjecture by Jie Gao, Michael Langberg, and Leonard Schulman

I will start with a Helly-type conjecture by Gao, Landberg and Schulman:

For a convex set K in $\mathbb{R}^d$ an ϵ enlargement of K is $K+\epsilon(K-K)$. (Where $K-K=\{x-y : x_0, y \in K\}$.)

Conjecture 15 *For every d, k and ϵ there is some $h = h(d,k,\epsilon)$ with the following property. Let $\mathcal{F}$ be a family of unions of k convex sets. Let $\mathcal{F}^\epsilon$ be the family obtained by enlarging all the involved convex sets by ϵ.*

If every h members of $\mathcal{F}$ have a point in common then all members of $\mathcal{F}^\epsilon$ have a point in common.

2.2 A new exciting topological Helly

Let me mention a recent exciting Helly-type theorem by Xavier Goaoc, Pavel Paták, Zuzana Safernová, Martin Tancer and Uli Wagner.

Theorem 2 *For every $\gamma > 0$ there is $h(\gamma, d)$ with the following property: Let $\mathcal{U}$ be a family of sets in $\mathbb{R}^d$. Suppose that for every intersection L of m members of $\mathcal{U}$ and every $i \le [d-1/2]$, we have$b_i(L) \le \gamma$. Then if every $h(\gamma, d)$ members of $\mathcal{U}$ have a point in common there is a point in common to all sets in $\mathcal{U}$.*

2.3 Fractional Helly

We will mention here two conjectures regarding the fractional Helly property and two related theorems. A class of simplicial complexes is hereditary if it is closed under induced subcomplexes. For a simplicial complex K, $f_i(K)$ is the number of i-faces of K, $b(K)$ is the sum of (reduced) Betti numbers of K.

Conjecture 16 (Kalai and Meshulam) *Let $C > 0$ be a positive number. Let $\mathcal{K}$ be the hereditary family of simplicial complexes defined by the property that for every simplicial complex $K \in \mathcal{K}$ with n vertices,*

$$b(K) \le Cn^d.$$

Then for every $\alpha > 0$ there is $\beta = \beta(d, C) > 0$, with the following property. For $K \in \mathcal{K}$, Then if $f_d(K) \geq \alpha\binom{n}{d+1}$ then $\dim(K) \geq \beta \cdot n$.

The conclusion of the conjecture is referred to as the fractional Helly property of degree d.

Conjecture 17 *Let $\mathcal{U}$ be a family of sets in $\mathbb{R}^d$. Suppose that for every intersection L of m members of $\mathcal{K}$, $b(L) \leq \gamma M^{d+1}$. Then $\mathcal{U}$ satisfies a fractional Helly property of order d.*

Theorem 3 (Bárány and Matousek) *Families of integral points in convex sets in $\mathbb{R}^d$ satisfies a fractional Helly property of order d.*

Theorem 4 (Matousek) *Families of sets of bounded VC-dimension in $\mathbb{R}^d$ satisfies a fractional Helly property of order d.*

Problem 18 *Does Radon theorem imply the fractional Helly property?*

3 Two questions by Imre on convex polytopes

Problem 19 *Let P be a d-polytope Is $f_k(P) \geq \max f_0(P), f_{d-1}(P)$?*

Problem 20 *Is there, for every positive integer d, a constant c_d such that the number of maximal flags of faces for a d-polytope is at most by c_d times the total number of faces (of all dimensions) of P?*

Bárány's first question must be correct (or, so I think, most of the times) but we cannot prove it. It falls into a much more general statement called *the generalized upper bound theorem*.

Conjecture 21 (Generalized upper bound conjecture) *Let K be a polyhedral complex that can be embedded into $\mathbb{R}^d$, let C be the boundary complex of a cyclic d-polytope. Then if $f_i(K) \leq f_i(C)$ for some $i \geq 0$ it follows that $f_j(K) \leq f_j(C)$ for every $j \geq i$.*

Bárány's second question is very mysterious and I don't know what the answer should be even for $d = 5$. I would guess that for polyhedral spheres the answer is negative but this remains open as well.

Happy birthday, dear Imre