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NEIGHBORLY FAMILIES OF 2^d d-SIMPLICES IN E^d

The purpose of this note is to prove the following

THEOREM 1. *There exists a neighborly family of* 2^d *d-simplices in* E^d, *for all* d, $d \geqslant 4$.

A family of d-simplices in the Euclidean d-space E^d is called neighborly [1–9] if every pair of them have a $(d-1)$-dimensional intersection. Let $f(d)$ denote the maximum number of d-simplices in a neighborly family in E^d. Begemihl [1] proved that $8 \leqslant f(3) \leqslant 17$, and conjectured that $f(3) = 8$; Baston [2] proved that $8 \leqslant f(3) \leqslant 9$. It had been repeatedly conjectured [2, 4, 5, 6] that $f(d) = 2^d$ for all d, $d \geqslant 3$, yet as mentioned in [9] even $f(d) \geqslant 2^d$ has not been shown for any d, $d \geqslant 4$. In [9] we proved that $f(d) \leqslant (2/3)(d+1)!$ holds for all d, $d \geqslant 4$. In this note we establish $f(d) \geqslant 2^d$ for all d, $d \geqslant 4$.

Theorem 1 clearly follows from

THEOREM 2. *For every* $d, d \geqslant 2$, *there exists a neighborly family of* 2^d *d-simplices in* E^d *and a line that meets the interiors of all of the d-simplices.*

Proof of Theorem 2. The proof is by induction on d, starting with the case $d = 2$, treated easily as shown in Figure 1.

Assume inductively that E^d contains a neighborly family of 2^d d-simplices $P_1, \ldots, P_n (n = 2^d)$ and that there is a line L in E^d for which $L \cap \text{Int } P_i \neq \varnothing$ holds for all i, $1 \leqslant i \leqslant n$. For every i, $1 \leqslant i \leqslant n$, let $y_i \in L \cap \text{Int } P_i$ and let $e_i > 0$ be chosen so that $B(y_i, e_i) \subset \text{Int } P_i$, where $B(y, e)$ is an e-ball centered at y; put $e = \min_i e_i$; clearly $e > 0$.

Let φ be the affine transformation of E^d which is obtained by a rigid motion taking the line L to the x_1-axis, followed by a translation and stretching along

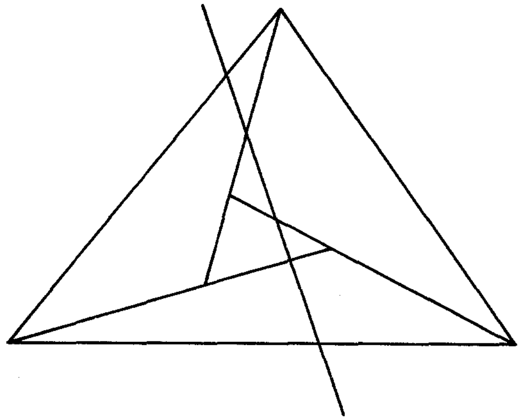

Fig. 1

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the x_1-axis so that $L \cap (\cup_i P_i)$ is taken into $\{(x_1,\ldots,x_d) | -1 \leqslant x_1 \leqslant 1\}$, then finally enlarge all the x_j, $2 \leqslant j \leqslant d$, by the factor $\sqrt{d}/e$.

It easily follows that φ is non-degenerate, hence $\varphi(P_1),\ldots,\varphi(P_n)$ is a neighborly family of d-simplices; in addition, the intersection of the hyperplane given by x_1 equals the first coordinate of $\varphi(y_i)$ with the unit cube $C = \{(x_1,\ldots,x_d) | -1 \leqslant x_i \leqslant 1 \text{ for all } i\}$ is in the interior of $\varphi(P_i)$, because of the enlarging of all x_j, $2 \leqslant j \leqslant d$, by a sufficiently large factor.

Let ψ be the rotation of E^d, given by just the interchanging of the coordinates x_1 and x_2. Clearly, $\psi(\varphi(P_1)),\ldots,\psi(\varphi(P_n))$ is another neighborly family of d-simplices, and for all i and j, $1 \leqslant i, j \leqslant n$, $\varphi(P_i) \cap \psi(\varphi(P_j))$ contains a neighborhood of the $(d-2)$-subcube $\{(\varphi(y_i), \psi(\varphi(y_j)), x_3,\ldots,x_d) | -1 \leqslant x_k \leqslant 1, 3 \leqslant k \leqslant d\}$, hence is d-dimensional.

Consider E^d as a subspace of E^{d+1} in the usual way, let M be the line through $(1,1,0,\ldots,0)$ in E^{d+1} in the direction of $(0,\ldots,0,1)$; let $A = (2,2,0,\ldots,0,2)$ and $B = (2,2,0,\ldots,0,-2)$; define Q_i, for $1 \leqslant i \leqslant 2n (= 2^{d+1})$, by

$$Q_i = \begin{cases} \operatorname{conv}(A \cup \varphi(P_i)) & 1 \leqslant i \leqslant n \\ \operatorname{conv}(B \cup \psi(\varphi(P_{n-i}))) & n+1 \leqslant i \leqslant 2n. \end{cases}$$

Clearly Q_i is a $(d+1)$-simplex for all i, $1 \leqslant i \leqslant 2n$. The family $\{Q_i | 1 \leqslant i \leqslant 2n\}$ is neighborly for the following reasons: If i and j are such that $1 \leqslant i \leqslant n$ and $n+1 \leqslant j \leqslant 2n$, then $Q_i \cap Q_j = \varphi(P_i) \cap \psi(\varphi(P_j))$ which is d-dimensional in E^d; if $1 \leqslant i; j \leqslant n$, then $Q_i \cap Q_j = \operatorname{conv}\{A \cup (\varphi(P_i) \cap \varphi(P_j))\}$, and since $\{\varphi(P_i) | 1 \leqslant i \leqslant n\}$ is neighborly in E^d, it follows that $\varphi(P_i) \cap \varphi(P_j)$ is $(d-1)$-dimensional in E^d; since $A \in E^{d+1} - E^d$, it follows that $Q_i \cap Q_j$ is d-dimensional. The case $n+1 \leqslant i; j \leqslant 2n$ is similar to the last one, hence it is omitted.

For the induction to work, we need to show that there is a line (M) meeting the interiors of the $Q_i - s$: The point $(1, 1)$ in the $x_1 x_2$-plane lies in the interior of every triangle of the form $(2, 2)$, $(s, 1)$ and $(s, -1)$, for all s, $-1 < s < 1$, and it lies also in the interior of every triangle of the form $(2, 2)$, $(1, t)$ and $(-1, t)$, for all t, $-1 < t < 1$, see Figure 2.

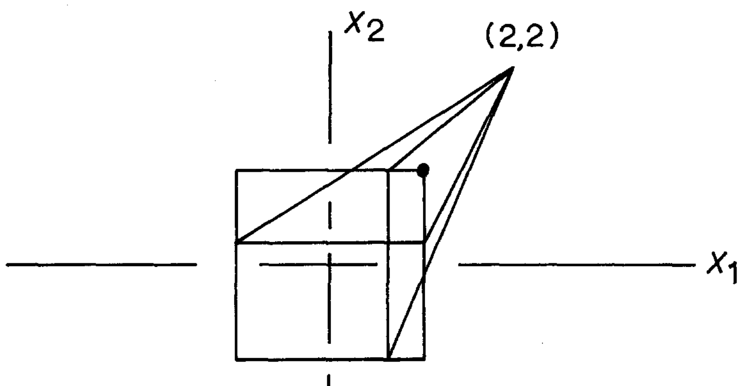

Fig. 2.

The line M' in the $x_1 x_2 x_{d+1}$-space passing through $(1, 1, 0)$ in the direction of $(0, 0, 1)$ intersects the interior of every triangle of the form $(2, 2, 2)$, $(s, 1, 0)$ and $(s, -1, 0)$ for all s, $-1 < s < 1$, and it also intersects the interior of every triangle of the form $(2, 2, -2)$, $(1, t, 0)$ and $(-1, t, 0)$ for all t, $-1 < t < 1$. The interior of $\varphi(P_i)$ contains a segment of the form $\{(s, t_i, 0, \ldots, 0) | -1 \leqslant s \leqslant 1\}$ in E^d, for some t_i, $-1 < t_i < 1$; the interior of $\psi(\varphi(P_i))$ contains a segment of the form $\{(s_i, t, 0, \ldots, 0) | -1 \leqslant t \leqslant 1\}$ in E^d for some s_i, $-1 < s_i < 1$, for every i, $1 \leqslant i \leqslant n$. It follows, therefore, that $M \cap \operatorname{Int} Q_i \neq \varnothing$ holds for all i, $1 \leqslant i \leqslant 2n$.

This completes the proof of Theorem 2.

Remark. It should be noted that Bagemihl's [1] idea has no simple extension to higher dimensions, because of the need to take two families of neighborly simplices in a hyperplane so that each one of the first family meets properly every one of the other family. One may think of taking an ε-perturbance of one neighborly family as the second one, but one simplex of the first family might be ε-pushed away from some simplex of the other.

Added in proof: M. Perles proved that $f(d) \leqslant 2^{d+1}$.

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