Remarks On

More about $A_n(x,y;p,q)$ and $F_n(x,y;p,q)$

by Gil Kalai

1. To stand alone, the paper needs a better title, perhaps, A Note on The Evaluation of Abel Sums.

2. The $F_n(x,y;r,s)$ may be called Associated Abel Sums and are preferably introduced through equation (3) modified as follows

$$A_n(x,y;p,q) = \sum_0^n \binom{n}{k}(k+x)^{k+p}(n-k+y)^{n-k+q}$$

$$= \sum_0 \binom{n}{k}(k+x)^{k+p}(n-k+y)^{q} \sum_0^{n-k}\binom{n-k}{j}(-k-x)^{n-k-j}(n+x+y)^j$$

$$= \sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\binom{n-k}{j}(-1)^{n-k-j}(k+x)^{p+n-j}(n-k+y)^q(n+x+y)^j$$

$$= \sum_{j=0}^{n}\sum_{k=0}^{n-j}\binom{n}{j}\binom{n-j}{k}(-1)^{n-k-j}(k+x)^{p+n-j}(n-k+y)^q(n+x+y)^j$$

$$= \sum_0^n \binom{n}{j}(n+x+y)^j F_{n-j}(x, y+j; p+n-j, q)$$

$$= \sum_0^n \binom{n}{j}(n+x+y)^{n-j} F_j(x, n-j+y; p+j, q)$$

where $F_n(x,y;r,s) = \sum_0^n \binom{n}{k}(-1)^{n-k}(k+x)^r(n-k+y)^s$. This differs from equation (3) in using the identity $\binom{n}{k}\binom{n-k}{j} = \binom{n}{j}\binom{n-j}{k}$ and inverting the order of summation.

3. The two relations $A_n(x,y;p,q) = A_n(y,x;q,p)$

$F_n(x,y;r,s) = (-1)^n F_n(y,x;s,r)$

imply the alternate relation

$$A_n(x,y;p,q) = \sum_0^n \binom{n}{j}(n+x+y)^{n-j} F_j(n-j+x, y; p, j+q)(-1)^j$$

4. $F_n(x,y;r,s)$ may be given in compressed form by the following modification of (18)

$$F_n(x,y;r,s) = \sum_0^n \binom{n}{k}(-1)^{n-k}(k+x)^r(n-k+y)^s$$

$$= \sum_0 \binom{n}{k}(-1)^{n-k} E_1^k x^r \; E_2^{n-k} y^s$$

$$= (E_1 - E_2)^n x^r y^s = (\Delta_1 - \Delta_2)^n x^r y^s$$

E_1 is the shift operator, operating on x: $E_1 x = 1+x$, E_2 is the shift operator for y.

This implies the instances

$$F_n(x,y;r,0) = \Delta^n x^r$$

$$F_n(x,y;r,1) = y\Delta^n x^r - n\Delta^{n-1}x^r$$

$$F_n(x,y;r,2) = y^2\Delta^n x^r - n(2y+1)\Delta^{n-1}x^r + 2\binom{n}{2}\Delta^{n-2}x^r$$

From my book Combinatorial Identities, p. 203

$$\Delta^n x^r = n! \sum_{j=0} \binom{r}{j} x^{r-j} S(j,n) \quad - S(j,n) = S_2(j,n)$$

= Stirling no. of second kind

$$= \sum_0^x \binom{x}{k}(n+k)!\, S(r, n+k)$$

$$= (n+x)\Delta^n x^{r-1} + n\Delta^{n-1}x^{r-1} \quad \text{- p. 204 (corrected)}$$

5. The Abel Sum evaluations provided by the above: first

$$A_n(x,y;p,0) = \sum_0 \binom{n}{k}(n+x+y)^{n-k}\Delta^k x^{p+k}$$

which holds for all p, positive or negative, is a striking simplification of the results in the table in my book. Indeed the case $p=1$, which is the Abel result, provides probably the simplest derivation. The next

$$A_n(x,y;p,1) = \sum_0 \binom{n}{k} (n+x+y)^{n-k} \left[(n-k+y)\Delta^k x^{k+p} - k\Delta^{k-1} x^{k+p} \right]$$
$$= \sum_0 \binom{n}{k} (n+x+y)^{n-k} \left[(n+y)\Delta^k x^{k+p} - k\Delta^{k-1} (x+1)^{k+p} \right]$$

[The second form also follows from the recurrence

$$F_n(x,y;r,s) = (n+y)F_n(x,y;r,s-1) - nF_{n-1}(x+1,y;r,s-1)]$$

is only a little less impressive.

6. I am glad you have noticed the inverse to (3), but really nothing is done with it. At the least, it provides an identity, every time an A_n is evaluated. Many of these may not be worth exhibiting but one at least should be noted.

7. The identities on p. 4 seem distracting in the present paper. Perhaps they fit better in the companion paper you mention in your letter (I would like to see that paper!). The extensions on p. 5 should be left to a later paper, where you can do justice to them. I prefer $A_n(x,y;a,b;p,q)$ to $A_n^{(a,b)}(x,y;p,q)$ (makes life easier for typists and typesetters). Possibly you need to go to $B_n(x,y;a,b;p,q)$ for one or other of the two extensions in (25) and (26).

8. As to format, I think it is nicer to use i, j, k for the sum variables and keep p, q, r, s for exclusive use as sum parameters

John Riordan Sept 25, 1972