Remarks on
A note on an evaluation of Abel Sums
by Gill Kalai (June, 1974)

1. The first page could be written with greater brevity and clarity as follows:

Following [1], the Abel Sums are taken as defined by

$$A_n(x,y;p,q) = \sum_{0}^{n} \binom{n}{k} (k+x)^{k+p} (n-k+y)^{n-k+q} \qquad (1)$$

The purpose of this note is to show the utility in their evaluation of auxiliary sums, here called Associated Abel Sums, and defined by

$$F_n(x,y;p,q) = \sum_{0}^{n} (-1)^{n-k} \binom{n}{k} (k+x)^{p} (n-k+y)^{q} \qquad (2)$$

Indeed, the two are interrelated by

$$A_n(x,y;p,q) = \sum_{0}^{n} \binom{n}{k} (n+x+y)^{k} F_{n-k}(x,y+k;p+n-k,q) \qquad (3)$$

$$F_n(x,y;p,q) = \sum_{0}^{n} (-1)^{k} \binom{n}{k} (n+x+y)^{k} A_{n-k}(x,y+k;p-n,q) \qquad (4)$$

(Note that (4) does not agree with your equation (12))

2. Equations (3) and (4) would be followed by their verifications; (3) is verified by equation (3) of ~~the~~ your present paper. (4) is verified by

$$F_n(x,y;p,q) = \sum_{0}^{n} (-1)^{k} \binom{n}{k} (n+x+y)^{k} \sum_{0}^{n-k} \binom{n-k}{j} (j+x)^{p-n+j} (n-j-k+y)^{q+n-j}$$

$$F_n(x,y;p,q) = \sum_{j=0}^{n} \sum_{k=0}^{n-j} \binom{n}{j}\binom{n-j}{k}(-1)^k (n+x+y)^k (j+x)^{p+n+j} (n-j+y)^{q+n-j-k}$$

$$= \sum_{j=0}^{n} \binom{n}{j} (j+x)^{p-n+j} (n-j+q)^q \sum_{0}^{n-j} \binom{n-j}{k} (-1)^k (n+x+y)^k (n-j+y)^{n-k-j}$$

$$= \sum_{j=0}^{n} \quad " \quad (-1)^{n-j} (j+x)^{n-j}$$

$$= \sum_{j=0} (-1)^{n-j} \binom{n}{j} (j+x)^p (n-j+y)^q$$

3. The alternative proof for (3) (equations (i)-(ii)) should be omitted for brevity.

4. The parameters p, q should be used throughout the papers (in place of r, s; z, t, etc in the present paper)

5. Because of your references to my book, I think you should use its notation for Stirling numbers; ~~ie~~ $S(j,n)$ instead of $S_2(j,n)$ ~~etc~~. Probably you have a special fondness for your notation, but it is an unnecessary burden for the reader in this paper.

6. Equation (19) on p.4 should be followed immediately by the results on page 5 for $p = -1, -2, 0, 1$. I would remark that the result for $p=-1$ ~~is probably the most~~ is a new and perhaps the simplest derivation of Abel's result ~~follows almost without calculation~~. Equation (21) seems unnecessary.

7. Nothing is done with equation (25). At least the two instances $p = 0, 1$ should be noticed; since $A_n(x,y;0,1) = A_n(y,x;1,0)$ the first is a verification. Either that or omit (25).

8 Both (26) and (27) are interesting results. Both can be abbreviated. For (26)

$$F_n(x,y;p,q) = \sum_0^n (-1)^{n-k} \binom{n}{k} (k+x)^p \sum_0^q \binom{q}{j} (-k-x)^{q-j} (n+x+y)^j$$

$$= \sum_0^n (-1)^{q-j} (n+x+y)^j \sum_0^n (-1)^{n-k} \binom{n}{k} (k+x)^{p+q-j}$$

$$= \sum_0^n (-1)^{q-j} (n+x+y)^j F_n(x,0;p+q-j,0)$$

Incidentally l is a symbol which for typesetters causes confusion with 1; I usually avoid it.

9. The equation numbers should be on the right hand side.

John Riordan
July 8, 1974