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The pendulum under vibrations revisited

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Abstract
A simple intuitive physical explanation is offered, of the stability of the inverted pendulum under fast violent vibrations. The direct description allows to analyze, both intuitively and rigorously, the effect of vibrations in similar, and in more general, situations. The rigorous derivations in the paper follow a singular perturbations model of mixed slow and fast dynamics. The approach allows applications beyond the classical inverted pendulum model.

Keywords: Stephenson–Kapitza pendulum, violent vibrations, vibrational forces, stabilization, Young measures
Mathematics Subject Classification numbers: 70K20, 70K05, 34C29, 34D20, 34E15.

1. Introduction

Back in 1908, Stephenson [20], showed that when fast vertical vibrations are applied to the point of suspension of an inverted pendulum, the otherwise unstable equilibrium becomes stable. In 1951, Kapitza [11, 12], rediscovered the phenomenon, provided a detailed analysis, and pointed out interesting applications. Such an inverted pendulum is often referred to as the Kapitza pendulum or the Stephenson–Kapitza pendulum. The results have ignited an abundance of theoretical investigations, and applications in many areas in science and engineering. In spite of its long history and research efforts, the phenomenon is still considered by many as counter-intuitive.

In the next section we offer a simple intuitive physical explanation of the stabilization effect. The explanation relates to the equation of motion only, and does not rely on any specific feature of the pendulum, thus it can be applied in a much wider scope. In section 3 we recall a singular perturbations model employing Young measures, that plays a role in the proof of the main result. The latter is stated and proved in section 4. An outline of a proof not relying on Young measures is also given. The stability of the inverted pendulum is examined in section 5. In section 6 we work out some examples. The type of vibrations analyzed in these sections assures that the limit dynamics of the pendulum solves a differential equation. In the closing section we offer some comments on extensions of the method, including a case, extensively studied in the literature, where the limit of the solutions does not solve an ordinary differential equation, yet, the stability can be established.

We now list (a non-exhaustive list indeed) some approaches available in the literature, to study the phenomenon. The differential equation that describes the inverted pendulum is (2.1), presented in the next section.

As mentioned, the first account of stability induced by vibrations is credited to Stephenson, who published several papers on the impact of forced oscillations, among them [20], where the stabilization of the otherwise unstable spring has been demonstrated. The equations analyzed in [20] are linear, and the mathematical technique involves power series analysis. The vibrations in Stephenson's analysis are of the form $n^2a(nt)$ where $n \to \infty$ (thus n corresponds to our $\frac{1}{\varepsilon}$). The term n^2 does not fit the framework of our rigorous analysis; we allude to that in the closing section.

A later account is due to Kapitza [11, 12], where, along with the theory, experiments validating the mathematical claims are reported (and photos are included). The analyzed equation is that of the inverted pendulum. The analysis applies to small oscillations, and it is based on approximation via averaging. Also here, the vibrations take the form $m^2a(mt)$. Kapitza coined the term *vibrational momentum* as the cause of the stabilization of the upward position. The Kapitza's method is echoed in many contributions, e.g., in Landau and Lifshitz [13, section 30].

An approach based on the method of averaging has been developed and applied to a wide spectrum of applications. See Bogolyubov and Mitropolskii [4], Volosov [21], and Mitropolskii and Dao [19]. The approach requires sophisticated changes of variables (that at times blur the physics behind the phenomenon). Further applications, in a wide scope of areas, of stabilization via vibrations were derived using the method of averaging.

Other interesting approaches for the analysis of the phenomenon, and attempts to explain it, were offered. We mention Blitzer [3] who considered vibrations in several directions. Butikov [5], [6, chapter 10], offered an explanation that focuses on the torque generated by the vibrations. Kalmus [10], examined vibrations generated by piecewise constant inputs, and carefully followed the resulting dynamics. Using explicit solutions, Arnold [1, section 25 E] showed that vibrations formed by alternating constant forces, stabilize the inverted pendulum.

An elegant and fruitful approach to verify, and explain, the stabilization phenomenon has been developed by Mark Levi in a series of papers. The method not only explains the stabilization of the inverted pendulum, but also ties this phenomenon to other physical phenomena. See Levi [14–16], Levi and Weckesser [18], and see Levi [17] for an overview that includes applications to, and interpretations of, other physical systems driven by vibrations. Levi's approach is based on a careful averaging of the geometry of the solutions, in particular, movements along the tractrix, inducing centrifugal forces that stabilize the pendulum. Levi's geometrical approach has been followed by many, improving the stability parameters and establishing properties related to stability, for instance, the existence of periodic solutions. We mention here Csizmadia and Hatvani [7, 8].

A simple proof, not alluding directly to the physics behind the phenomenon, is offered in Evans and Zhang [9]. Some technical steps that we use in this paper are based on tools introduced in [9].

It should be mentioned that videos documenting the vibrational stabilization effect, can be found on the internet. While the demonstrations agree with the mathematics, the interpretations, at times, deviate from it. We comment on some relations between what we see and the mathematics, in the body of the text.

One may argue, and rightly so, that any mathematical proof carries with it an explanation. Since simplicity is in the eye of the beholder, we do not get here into comparisons between the various explanations and the one offered next.

2. An explanation of the stabilization phenomenon

First we display the differential equation that governs the movement of the Stephenson–Kapitza pendulum. The derivation of the equation is standard. For a complete and elegant justification see Levi [17]. See also the classical textbooks of Arnold [1, section 25] and Landau and Lifshitz [13, section 30].

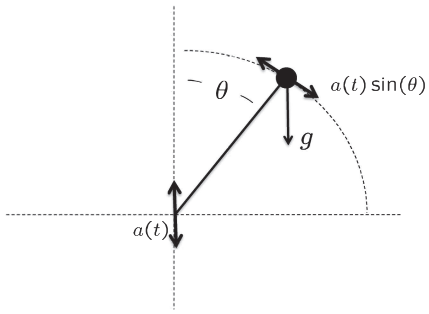

Figure 1.

Consider (see figure 1) a weightless rod of length l (for simplicity we take $l = 1$), with a point mass at its end, subject to a constant gravitational force g. In addition, the suspension point, or pivot, of the pendulum is subject to time varying acceleration $a(t)$ in the vertical direction. We denote by θ the angle between the vertical direction and the direction of the rod. Compatible with θ, we agree that downward is the positive direction for the pivot. The acceleration input $a(t)$ is autonomous, namely, it is caused by an outside force, and not as a feedback of the position or the velocity of the point mass. The velocity of the pivot is denoted by $v(t)$, it is an integral of the acceleration. We assume the following.

Assumption 2.1. (1) The acceleration function $a = a(\cdot)$ is integrable, periodic with, say, period 1. (2) The average of the acceleration over a period satisfies $\int_0^1 a(s)\mathrm{d}s = 0$. (3) The velocity $v(s)$ satisfies $\int_0^1 v(s)\mathrm{d}s = 0$.

The three conditions reflect a situation where at the end of one period, the pivot of the pendulum returns to its initial position. Indeed, condition (2) implies that $v(\cdot)$ is one-periodic, and condition (3) assures that among all the integrals of the acceleration, the one we use makes the location of the pivot one-periodic as well.

The high frequency high amplitude vibrations, at times referred to as *violent*, that we address here are modeled by $\frac{1}{\varepsilon}a(\frac{t}{\varepsilon})$, with ε a small positive parameter. The angle $\theta(t)$ satisfies then the second order ordinary differential equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = (g + \frac{1}{\varepsilon}a(\frac{t}{\varepsilon}))\sin(\theta). \tag{2.1}$$

The explanation, and the rigorous derivations, that follow, do not rely on physical properties specific to the pendulum. They apply to a right-hand side more general than (2.1) (see (4.1) below). Here we display a particular case, suffices to explain the stabilization phenomenon.

$$\frac{d^2x}{dt^2} = \frac{1}{\varepsilon}a(\frac{t}{\varepsilon})f(x), \tag{2.2}$$

with $f(x)$ being, say, positive and continuously differentiable, with its derivative $f'(x)$ being, say, positive. In the rest of this discussion we allude to (2.2).

Here is our simple, and physically transparent, explanation of the phenomenon. It is not presented on a level of a rigorous proof (although it can probably be refined to make one).

Intuitive explanation 2.2. *The discussion here is of the local forces acted on the mass x. The implications to the stability are displayed in section 5.*

While the given acceleration function $a(t)$ satisfies assumption 2.1, the position x is driven by a different acceleration, namely, $a(t)f(x(t))$. We refer to the latter as the 'effective acceleration'. We view $f(x(t))$ as a coefficient that modifies the force exerted by the acceleration $a(t)$. We refer to $f(x(t))$ as the 'efficacy curve'. In particular, different efficacy curves may result in different dynamics.

We claim that there is a significant difference between the efficacy curves when the acceleration is positive (i.e., $a > 0$), and when a negative acceleration is in action. The consequence of this difference is that when the efficacy curve $f(x(t))$ is positive and increasing, the impact of the negative acceleration is greater than that of the positive acceleration.

There are two complementary differences between the negative and positive accelerations in (2.2). Both differences are associated with the geometry of the efficacy curves. We examine the two differences separately.

First, we claim, the efficacy curve generated by a negative acceleration is more concave than the efficacy curve generated by a positive acceleration. We shall verify this momentarily, and now explain why the degree of concavity matters. Consider the following schematic situation. A unit mass is pushed during a time interval, using a constant acceleration a, say positive. Compare now the results of moving the mass under two efficacy curves, one is convex, the other is concave, as in figure 2. Other than that, the efficacy curves share the same boundaries.

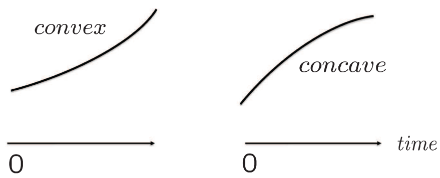

Figure 2.

The velocity added to the mass during the time interval is the integral of the effective acceleration, namely the acceleration times the efficacy curve. It is clearly larger with the concave curve. In particular, the velocity added to the mass when the efficacy curve is more concave, is larger than the velocity added to the mass when the efficacy curve is more convex. We now verify that when the acceleration a is negative the efficacy curve is more concave, than when a is positive. To that end recall that the degree of concavity is determined by the second derivative of the curve. We compute now this second derivative at a given point x_0, say at $t = t_0$, when the acceleration is a.

$$\frac{\mathrm{d}^2 f(x(t))}{\mathrm{d}t^2}(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}(f'(x_0)\frac{\mathrm{d}x}{\mathrm{d}t}(t_0))$$
$$= f''(x_0)(\frac{\mathrm{d}x}{\mathrm{d}t}(t_0))^2 + af'(x_0)f(x_0). \tag{2.3}$$

It is clear that the second derivative of the efficacy curve $f(x(t))$ at the time t_0 is larger when a is positive, than when a is negative.

The implication to the solution of the differential equation is clear. If we apply positive acceleration a and negative acceleration $-a$, over two consecutive equal periods, the net effect at the end of the round is to render the mass a smaller velocity (recall that in the pendulum case, negative velocity reflects upward movement).

Smaller velocity after a period may not yield smaller position. Indeed, the displacement of the mass during a time interval, is the integral of the velocity, namely the second integral of the acceleration times the efficacy curve. Here is the second difference between the efficacy curves of negative and positive accelerations.

We claim that the efficacy curve generated by the negative acceleration is decreasing faster than the efficacy curve generated by a positive acceleration. We shall justify this momentarily, but first explain why the difference between the slopes matters. Consider again a schematic situation. A unit mass is pushed during a time interval, using a constant acceleration a, say positive. Compare now the results of moving the mass under two efficacy curves, one is increasing, the other is decreasing, as in figure 3. Other than that, the efficacy curves share the same boundaries.

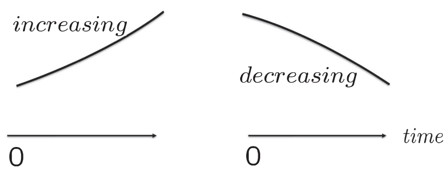

Figure 3.

The velocity of the mass in the decreasing case is accumulated, i.e., reaching a high level, faster than in the increasing one. Therefore, the integral of the velocity, namely the displacement, is larger when the efficacy curve is decreasing faster. A simple computation, say when the curves are linear, would verify the claim. To verify the difference in the slopes of the efficacy curves when a is positive and negative, is straightforward. Indeed, the force exerted by the acceleration pushes x down or up according to a being negative or positive. Since $f'(x)$ is positive, the efficacy curve $f(x(t))$ decreases faster when a is negative.

Summing up our intuitive explanation: the efficacy curve (which in our case is, in fact, generated by the acceleration) alters the force exerted on the mass x, from a force satisfying the conditions in assumption 2.1, to one which exhibits differences between the negative and positive accelerations. Since the efficacy curve when a is negative is more concave and decreases faster than the efficacy curve when a is positive, the additional velocity and displacement over, say, one period, are not zero as in assumption 2.1, but negative. Namely, the effective acceleration shifts the mass to velocity and position smaller than they would be without applying the acceleration. (Clearly, analogous arguments are valid when $f(x)$ or $f'(x)$ may be negative).

The explanation reflects the situation that a round of applying the acceleration $a(t)$, yields access of effective acceleration, in the negative direction. Modifying Kapitza's terminology, we may refer to this access as the 'vibrational force'.

Still, there are gaps between the intuition just displayed and the general case of the pendulum, or the general case of (2.2). First, unlike the schematic examples, the deceleration function (namely, when $a(t)$ is negative), is not the negative of the acceleration function, as all what we know is what assumption 2.1 guarantees. Second, the values $f(x)$ and $f'(x)$ may change during the time interval, which makes the concavity comparisons difficult. Third, difference between positive and negative accelerations becomes negligible over the very short intervals, of length ε, during which the changes occur.

What bridges the gaps is the limit process. Indeed, on small intervals $f(x(t))$ and $f'(x(t))$ are almost constant, and the violence component makes sure that even along small intervals the vibrational force does not become negligible. Furthermore, conditions (2) and (3) in assumption 2.1 assure that the total positive force, is equal to the negative force. Therefore, the summation of the excess forces will be in favor of negative acceleration.

To demonstrate how the limit process bridges the aforementioned gaps, we display the following computations. In what follows we denote $\frac{\mathrm{d}x_\varepsilon}{\mathrm{d}t} = y_\varepsilon$ and denote by $l(t)$ the location of the pivot. Also, assume that we start at time t_0 where the velocity $v(\frac{t_0}{\varepsilon})$ of the pivot is equal to 0.

$$
\begin{aligned}
y_\varepsilon(t_0+\varepsilon) - y_\varepsilon(t_0) &= \int_{t_0}^{t_0+\varepsilon} \frac{1}{\varepsilon} a(\frac{t}{\varepsilon}) f(x_\varepsilon(t))\mathrm{d}t = -\int_{t_0}^{t_0+\varepsilon} v(\frac{t}{\varepsilon}) f'(x_\varepsilon(t)) y_\varepsilon(t)\mathrm{d}t \\
&\approx f'(x_\varepsilon(t_0)) \int_{t_0}^{t_0+\varepsilon} v(\frac{t}{\varepsilon}) y_\varepsilon(t)\mathrm{d}t \approx f'(x_\varepsilon(t_0)) \int_{t_0}^{t_0+\varepsilon} \varepsilon l(\frac{t}{\varepsilon}) \frac{\mathrm{d}y_\varepsilon}{\mathrm{d}t}(t)\mathrm{d}t \\
&= f'(x_\varepsilon(t_0)) \int_{t_0}^{t_0+\varepsilon} l(\frac{t}{\varepsilon}) a(\frac{t}{\varepsilon}) f(x_\varepsilon(t))\mathrm{d}t \\
&\approx f'(x_\varepsilon(t_0)) f(x_\varepsilon(t_0)) \int_{t_0}^{t_0+\varepsilon} l(\frac{t}{\varepsilon}) a(\frac{t}{\varepsilon})\mathrm{d}t \\
&= -f'(x_\varepsilon(t_0)) f(x_\varepsilon(t_0)) \int_{t_0}^{t_0+\varepsilon} v(\frac{t}{\varepsilon}) v(\frac{t}{\varepsilon})\mathrm{d}t = -\varepsilon \hat{V} f'(x_\varepsilon(t_0)) f(x_\varepsilon(t_0)), \quad (2.4)
\end{aligned}
$$

where $\hat{V} = \int_0^1 v(t)^2 \mathrm{d}t$. Here the symbol $\approx$ stands for an approximation of order $o(\varepsilon)$ (little oh of ε, namely, an error that converges to 0 even if divided by ε).

Indeed, we see that while after one round of the vibrations the pivot of the pendulum, or of any generator of the vibrations satisfying assumption 2.1, returns to its original position, the velocity driven by the differential equation is in a different position.

Here is the justification of the derivations in (2.4). The top equality holds since the added velocity of the mass is the integral of the effective acceleration. The next equality is obtained via integration by parts, and the choice that $v(\cdot)$ at the two ends of the interval is equal to 0. Since $y_\varepsilon(t)$ appears in the integrand, we also derive from this equality that $y_\varepsilon(t_0+\varepsilon) - y_\varepsilon(t_0)$ tends to 0 as $\varepsilon \to 0$. The next approximation is obtained by observing that the change of $f'(x(t))$ over the interval is small when ε is small, and that the integration is on a segment of length ε. The next line is obtained by integration by parts, and the approximation is justified since, as we just showed, $y_\varepsilon(t_0+\varepsilon) - y_\varepsilon(t_0) \to 0$ as $\varepsilon \to 0$, and since $l(\cdot)$ is one-periodic. Next equality follows from (2.2). Taking $f(x_\varepsilon(t_0))$ out of the integral results, as was the case with $f'(x_\varepsilon(t_0))$, in an $o(\varepsilon)$ error. Next equality is obtained via integration by parts and observing that both $l(t)$ and $v(t)$ are one-periodic. The last equality follows from a change of the time scale.

All in all, we get that $y_\varepsilon(t_0+\varepsilon) - y_\varepsilon(t_0)$ is equal to $-\varepsilon \hat{V} f'(x_\varepsilon(t_0)) f(x_\varepsilon(t_0))$, up to $o(\varepsilon)$ order. Thus, the change in the velocity over an ε-period is driven by a vibrational force of magnitude

$\hat{V} f'(x_\varepsilon(t_0)) f(x_\varepsilon(t_0))$. The latter expression is well known in the literature, and will show up in the rigorous derivations below.

Notice, however, that (2.4) refers to exactly one period. During the period, $y_\varepsilon(\cdot)$ may exhibit large oscillations, due to the violent vibrations. One may continue the analysis by taking the weak limit of $y_\varepsilon(\cdot)$ and provide a rigorous proof of the limit behaviour of equation (2.2). We do not follow this avenue as later in the paper we verify, rigorously, a more general result.

3. Mixed slow and fast dynamics

We recall here the model analyzed in [2], and display results that play a role in the analysis that follows. The framework of the model is more general than what is actually needed for the analysis of the vibrating pendulum. The general model is displayed here as a tool for understanding better the limit structure of solutions to equations with vibrating elements, and for further applications, one is displayed in the closing section. The mathematical notions we use are standard.

The ordinary differential equations we examine are of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = G(x) + \frac{1}{\varepsilon} F(x), \tag{3.1}$$

with x in R^n, the n-dimensional Euclidean space, and when $\varepsilon > 0$ is small. The model is referred to as *mixed slow and fast dynamics* (in contrast to the particular case of *coupled slow and fast variables* as in the Tikhonov singular perturbations model). We are interested in the limit behavior of solutions to (3.1) as $\varepsilon \to 0$.

The equation

$$\frac{\mathrm{d}x}{\mathrm{d}s} = F(x), \tag{3.2}$$

is the *fast equation* associated with (3.1). Notice the change of time scales between (3.1) and (3.2), namely, $t = \varepsilon s$. In particular, the fast equation does not depend on the small parameter ε.

Solutions to the differential equation (3.1) may exhibit fast oscillations when ε is small. Thus the limit, as $\varepsilon \to 0$, of such solutions may not be represented as a function. We claim that under quite natural conditions the following holds.

$$\begin{aligned} &\textit{A limit, as } \varepsilon \to 0, \textit{ of solutions to } (3.1), \textit{ can be described as an invariant} \\ &\textit{measure of the fast equation } (3.2), \textit{ drifted by the slow component } G(x). \end{aligned} \tag{3.3}$$

In order to put (3.3) within a rigorous mathematical framework, we need the notions of a *probability measure* on R^n. The space of probability measures is endowed with the *weak convergence of measures*, which is metrizable, say with the Prohorov metric. We consider then measurable mappings defined on a time interval, say $[0, T]$, with values being probability measures on R^n. Such a mapping is called a *Young measure*. Via integration over time, of the values of a Young measure, the Young measure can be interpreted as a measure on $[0, T] \times R^n$ (it will be a probability measure if $T = 1$). Convergence among Young measures is then taken as the weak convergence of the integrated measures. A useful property of this convergence is that integration of bounded continuous functions with respect to the converging Young measures, is a continuous operation.

A point-valued function $x(t) : [0, T] \to R^n$ can be interpreted as a Young measure whose values are Dirac measures, each supported on $\{x(t)\}$. In particular, a sequence $x_j(\cdot)$ of such mappings may converge in the sense of Young measures to a Young measure whose values may not be supported on singletons. An intuitive interpretation of such a convergence is that the distributions of the values of the elements $x_j(\cdot)$, converge to the distribution on $[0, T] \times R^n$ of the limit Young measure. This is the meaning of the displayed claim (3.3), namely, solutions $x_\varepsilon(\cdot)$ to (3.1) converge, in the sense of Young measures, to a probability measure-valued map, whose values are invariant measures of the fast equation.

Here are useful assumptions concerning (3.1).

Assumption 3.1. (i) $G(\cdot)$ and $F(\cdot)$ are continuous. (ii) The solutions $x_\varepsilon(\cdot)$ to (3.1) for $\varepsilon > 0$ small enough, emanating from initial conditions at $t = 0$ in a compact set, stay uniformly bounded on the finite time interval $[0, T]$. (iii) For every initial condition $x(0)$ the solution of the fast equation (3.2) is unique.

Assumption 3.2. For every initial condition $x \in R^n$, the ω-limit set $\omega(x)$, with respect to the fast dynamics (3.2), supports a unique invariant measure of (3.2), we denote this invariant measure by $\bar{\mu}(x)$.

The following result is the content of [2, theorem 4.4].

Theorem 3.3. *Under assumption 3.1, let $x_0 \in R^n$ and let $x_\varepsilon(\cdot)$ be solutions to (3.1), satisfying $x_\varepsilon(0) = x_0$, all defined on a common interval, say $[0, T]$. Then for every sequence $\varepsilon_j \to 0$ there exists a sub-sequence of $x_{\varepsilon_j}(\cdot)$ which converges in the sense of Young measures to a Young measure, say $\mu_0(\cdot)$, defined on $[0, T]$. The values of the limit Young measure are invariant measures of the fast equation (3.2). If, in addition, assumption 3.2 holds and $\bar{\mu}(x)$ is a Lipschitz map (into the space of probability measures), then $\mu_0(\cdot)$ is Lipschitz in the time variable.*

Given a Young measure limit $\mu(\cdot)$ of solutions to (3.1), we wish to know how its values progress in time. The way these invariant measures can be traced is via *slow observables*. While [2] examines general observables, in the applications that follow we need just the notion of *orthogonal observables*, namely, continuous real-valued functions on R^n that are first integrals of the fast equation, i.e., constant on trajectories of the fast equation. If $h(\cdot)$ is such an orthogonal observable, and if μ is an invariant measure of the fast dynamics supported on an ω-limit set of a solution to (3.2), the value $h(x)$ is constant on the support of μ, and we write then $h(\mu) = h(x)$ with x being any point in the support of μ.

The following result is verified in [2, theorem 6.5].

Theorem 3.4. *Under assumption 3.1, let $x_{\varepsilon_j}(\cdot)$ be solutions to (3.1) satisfying $x_{\varepsilon_j}(0) = x_0$ and defined on, say, $[0, T]$, and which converge, as $\varepsilon_j \to 0$, in the sense of Young measures, to $\mu_0(\cdot)$. Let $h(\cdot)$ be an orthogonal observable of the system. Then the measurements $h(x_{\varepsilon_j}(t))$ converge weakly on $[0, T]$ to $h_0(t) = h(\mu_0(t))$. Furthermore, if $h(x)$ is continuously differentiable in x, then $h_0(t)$ satisfies the differential equation*

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \int_{R^n} ((\nabla h)(x) \cdot G(x))\ \mu_0(t)(\mathrm{d}x). \tag{3.4}$$

If, in addition, assumption 3.2 holds and the mapping $\bar{\mu}(x)$ is Lipschitz, then the measurements $h(x_{\varepsilon_j}(t))$ converge uniformly, as $j \to \infty$, to $h_0(t)$.

Notice that (3.4) is not an ordinary differential equation, as the measure $\mu_0(t)$ is not given, and may not be determined by the observable $h(t)$. If it is determined by an orthogonal observable (or by several such observables), the equation becomes an ODE. In the general case one needs to employ ad hoc arguments in order to 'solve' the equation, as done in the next section.

4. A general result

Using the tools introduced in section 3, we establish a limit equation, as $\varepsilon \to 0$, for a general second order vibrational equation.

As was hinted earlier, a direct proof of convergence, not employing Young measures, is possible. See remark 4.2 below. We employ Young measures since they carry useful information, compatible with the explanation in section 2.

The differential equation of interest here is

$$\frac{d^2x}{dt^2} = g\left(x, \frac{dx}{dt}\right) + \frac{1}{\varepsilon}a\left(\frac{t}{\varepsilon}\right)f(x). \tag{4.1}$$

For convenience we denote $\frac{dx}{dt} = y$.

Theorem 4.1. *Suppose that $g(x, y)$ and $f(x)$ in (4.1) are continuously differentiable, and $a(\cdot)$ satisfies assumption 2.1. Suppose also that given a finite time interval $[0, T]$, solutions $(x_\varepsilon(t), y_\varepsilon(t))$ emanating from a common initial condition, are uniformly bounded. Let $x_\varepsilon(\cdot)$ be the unique solution to (4.1) with initial condition $(x(0), y(0))$ common to all ε. Then $x_\varepsilon(\cdot)$ converge, as $\varepsilon \to 0$, uniformly on $[0, T]$ to the solution $x(t)$ of*

$$\frac{d^2x}{dt^2} = \int_0^1 g(x, y + v(\sigma)f(x))d\sigma - \hat{V}f'(x)f(x), \tag{4.2}$$

with the same initial condition, and $y_\varepsilon(\cdot)$ converges weakly to $y(t)$. Here $\hat{V} = \int_0^1 v(s)^2 ds$.

Furthermore, the sequence $y_\varepsilon(\cdot)$ converges in the sense of Young measures to the Young measure whose value at $t \in [0, T]$ is the measure induced by the mapping that assigns to $\sigma \in [0, 1]$ the point $(x(t), y(t) + v(\sigma)f(x(t)))$.

First we display the equation as a first order system.

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= g(x, y) + \frac{1}{\varepsilon}a(\frac{t}{\varepsilon})f(x). \end{aligned} \tag{4.3}$$

Notice that the system is of a mixed slow and fast dynamics. Formally it is not of the type exhibited in (3.1), as the right-hand side of (4.3) depends on time. But since the acceleration $a(\cdot)$ is periodic, it is easy to convert the equation to a time-independent one (e.g., by introducing a variable, say τ, defined on a circle with circumference of length 1, that satisfies the equation $\frac{d\tau}{dt} = \frac{1}{\varepsilon}$, and let $a = a(\tau)$). We leave out these details and continue with (4.3) as is.

The fast system of (4.3) is

$$\begin{aligned} \frac{dx}{ds} &= 0 \\ \frac{dy}{ds} &= a(s)f(x). \end{aligned} \tag{4.4}$$

Given initial conditions (x_0, y_0), the unique solution to (4.4) is

$$\begin{aligned} x(s) &= x_0 \\ y(s) &= y_0 + v(s)f(x_0). \end{aligned} \tag{4.5}$$

The periodicity of $v(\cdot)$ implies that the invariant measures of (4.4) are the measures on the (x, y) space induced by the mapping that assigns to $\sigma \in [0, 1]$ the point $(x_0, y_0 + v(\sigma)f(x_0))$. This measure encompasses the limit behavior of the vibrations. In the sequel it will be convenient to employ the latter mapping, that induces the measure, rather than the measure itself.

It is easy to see that both assumptions 3.1 and 3.2 are satisfied, as well as the conditions displayed in theorem 3.3. Therefore, any limit Young measure is Lipschitz.

Now, notice that the state variable x is constant along solutions of the fast equation, hence it is an orthogonal observable. Its gradient as a measurement on R^2 is $(1, 0)$. In view of theorem 3.4, the observable x converges uniformly on any given finite interval, to the solution to the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \int_0^1 (y(t) + v(\sigma)f(x))\mathrm{d}\sigma. \tag{4.6}$$

As explained in the previous section, (4.6) is not an ordinary differential equation, since the term $y(t)$ in its right-hand side is not given or determined by x. Yet, we get useful information from theorem 3.3. First, sub-sequences of the functions $y_\varepsilon(\cdot)$, converge weakly to $y(t)$, and since the invariant measure derived from (4.5) is Lipschitz in (x, y), the limit functions $y(t)$ are Lipschitz, hence the limit solutions $x(t)$ of sub-sequences of solutions $x_\varepsilon(t)$ to (4.1), are continuously differentiable.

In order to reveal what $y(t)$ is, we integrate (4.1), and get

$$y_\varepsilon(t) = y_0 + \int_0^t g(x_\varepsilon(s), y_\varepsilon(s))\mathrm{d}s + \int_0^t \frac{1}{\varepsilon} a\left(\frac{s}{\varepsilon}\right) f(x_\varepsilon(s))\mathrm{d}s. \tag{4.7}$$

Integration by parts of the second integral when $\frac{t}{\varepsilon}$ is an integer (recalling that $v(0) = 0$ and that $v(\cdot)$ is one-periodic) yields

$$y_\varepsilon(t) = y_0 + \int_0^t g(x_\varepsilon(s), y_\varepsilon(s))\mathrm{d}s - \int_0^t v\left(\frac{s}{\varepsilon}\right) f'(x_\varepsilon(s))\, y_\varepsilon(s)\mathrm{d}s. \tag{4.8}$$

Consider the first integral in (4.8). Since $y_\varepsilon(s)$ converge, as $\varepsilon \to 0$, to the Young measure whose value at s is determined by the mapping that assigns to $\sigma \in [0.1]$ the value $y(s) + v(\sigma)f(x_0)$, the definition of the convergence in the sense of Young measures implies that the first integral converges to

$$\int_0^t \left(\int_0^1 g(x(s), y(s) + v(\sigma)f(x(s)))\mathrm{d}\sigma\right) \mathrm{d}s. \tag{4.9}$$

Consider now the term $v(\frac{s}{\varepsilon})y_\varepsilon(s)$ inside the second integral in (4.8). Since for a sub-sequence ε_j (for visual convenience we suppress in this paragraph the index j) the sequence $y_\varepsilon(s)$ converges to the Young measure whose value at s is determined by the mapping $\sigma \to y(s) + v(\sigma)f(x(s))$, defined on $[0, 1]$, the mapping $v(\frac{s}{\varepsilon})y_\varepsilon(s)$ converges to the Young measure whose value at s is determined by the mapping $\sigma \to y(s)v(\sigma) + v(\sigma)^2 f(x(s))$. Since the integral of $v(\cdot)$ is zero, the expectation (i.e., the average) of the latter measure is $f(x(s)) \int_0^1 v(\sigma)^2 \mathrm{d}\sigma$. As $\varepsilon \to 0$, and bearing in mind that the limit $y(s)$ is Lipschitz, and that the points t where $\frac{t}{\varepsilon}$ is an integer become dense in $[0, 1]$ as $\varepsilon \to 0$, it follows (together with (4.9)) that the limit as $\varepsilon \to 0$ of (4.8) is

$$y(t) = y_0 + \int_0^t \left(\int_0^1 g(x(s), y(s) + v(\sigma)f(x(s)))\mathrm{d}\sigma \right) \mathrm{d}s$$
$$- \int_0^t \left(\left(\int_0^1 v(\sigma)^2 \mathrm{d}\sigma \right) \right) f'(x(s))f(x(s))\mathrm{d}s. \tag{4.10}$$

Since $y(t)$ is the derivative of $x(t)$, we get that the limit trajectory solves the differential equation (4.2). To begin with, (4.6) may depend on a choice of a sequence ε_j for which the solutions converge to a Young measure. But since equation (4.2) does not depend on the choice of the sequence ε_j, it follows that solutions of (4.1) converge as $\varepsilon \to 0$ to solutions of (4.2).

The claims about the Young measure limit, and the weak limit, of $y_\varepsilon(\cdot)$ follow from the proof. This completes the proof.

Our interest in the Young measure limit stems from the observation that the values of the limit reflect the limit of the vibrations of the point mass, in line with (3.3). It is also useful in extensions of the model, as shown in example 7.2. The uniform convergence of $x_\varepsilon(\cdot)$ and the weak convergence of $y_\varepsilon(\cdot)$ in the previous theorem, can, however, be established while avoiding the Young measures vocabulary, verifying directly the consequences extracted from the properties of the Young measures convergence. A clever way of doing that is as follows.

Remark 4.2. Here is an outline of a direct proof of the uniform convergence of the state and the weak convergence of the velocity, stated in theorem 4.1. It was kindly suggested to me by Lawrence C Evans.

First, introduce an auxiliary variable, depending on ε,

$$q_\varepsilon = y_\varepsilon - v\left(\frac{t}{\varepsilon}\right) f(x_\varepsilon). \tag{4.11}$$

Differentiating (4.11), when $x_\varepsilon(\cdot)$ is a solution to (4.1), we get

$$\frac{\mathrm{d}q_\varepsilon}{\mathrm{d}t} = g(x_\varepsilon, y_\varepsilon) - v\left(\frac{t}{\varepsilon}\right) f'(x_\varepsilon)q_\varepsilon - v\left(\frac{t}{\varepsilon}\right)^2 f'(x_\varepsilon)f(x_\varepsilon). \tag{4.12}$$

Then $\frac{\mathrm{d}q_\varepsilon}{\mathrm{d}t}(t)$ is bounded. Hence a sub-sequence of $q_\varepsilon(\cdot)$ (we suppress the index in the sub-sequence) converges uniformly. From (4.11) we learn (since the weak limit of $v(\frac{t}{\varepsilon})$ is zero) that the uniform limit of $q_\varepsilon(\cdot)$ is $y(\cdot)$. Then $x_\varepsilon(\cdot)$ converges uniformly, say to $x(\cdot)$.

The definition of q_ε yields that

$$g(x_\varepsilon, y_\varepsilon) = g\left(x_\varepsilon, q_\varepsilon + v\left(\frac{t}{\varepsilon}\right) f(x_\varepsilon)\right). \tag{4.13}$$

The established convergences (of sub-sequences) imply now that the expression (4.13) converges weakly to the function whose value at time t is given by

$$\int_0^1 g(x(t), y(t) + v(\sigma)f(x(t)))\mathrm{d}\sigma. \tag{4.14}$$

Now, deriving (4.7), and (4.8), as in the previous proof, and replacing $y_\varepsilon(s)$ in the second term of (4.8), by $q_\varepsilon + v\left(\frac{t}{\varepsilon}\right) f(x_\varepsilon)$, as allowed by (4.11), yields

$$y_\varepsilon(t) = y_0 + \int_0^t g(x_\varepsilon(s), y_\varepsilon(s))\mathrm{d}s - \int_0^t v\left(\frac{s}{\varepsilon}\right) f'(x_\varepsilon(s)) \left(q_\varepsilon(s) + v\left(\frac{t}{\varepsilon}\right) f(x_\varepsilon(s)) \right) \mathrm{d}s. \tag{4.15}$$

Taking the previously established strong and weak limits (recalling that the weak limit of $v\left(\frac{t}{\varepsilon}\right)$ is zero) in (4.15) yields (4.10), hence (4.2). Since the latter does not depend on the choice of a sub-sequence, the uniform convergence and weak convergence hold, as claimed.

5. The Stephenson–Kapitza inverted pendulum

The equation governing the pendulum under vibrations, namely (2.1), is a particular case of the general case verified in the previous section. We now state the, well known, corresponding result, and then study the stability properties of the inverted pendulum. The time derivative of the position function $\theta(t)$ is denoted by $\eta(t)$.

Theorem 5.1. *Let initial conditions $\theta(0) = \theta_0$ and $\eta(0) = \eta_0$, and let a finite time interval $[0, T]$, be given. The solutions $\theta_\varepsilon(t)$ to (2.1) with the given initial conditions, converge uniformly on $[0, T]$ to the unique solution $\theta_0(t)$ to the equation*

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = (g - \hat{V}\ \cos(\theta))\sin(\theta), \tag{5.1}$$

with the same initial conditions; here $\hat{V} = \int_0^1 v(s)^2\mathrm{d}s$. The sequence $\eta_\varepsilon(t)$ converges weakly to $\eta(t)$. In fact, $\eta_\varepsilon(\cdot)$ converges in the sense of Young measures, to the Young measure whose value at time t is the measure induced by the mapping that assigns to $\sigma \in [0, 1]$ the point $(\theta(t), \eta(t) + v(\sigma)\sin(\theta(t)))$.

The preceding result identifies the vibrational force acting on the inverted pendulum under vibrations, namely, $-\hat{V}\ \cos(\theta)\sin(\theta)$. It enables to study its stability properties. Notice, however, that we study the stability of the limit equation. We comment later on the stability of the original equation when ε is small.

An apparent condition is that the position of the pendulum is within the region (if not empty)

$$\hat{V}\ \cos(\theta) > g. \tag{5.2}$$

Clearly, a small perturbation around the vertical position $(\theta, \eta) = (0, 0)$ will stay small. An initial condition (θ_0, η_0) in the region (5.2), with velocity $\eta > 0$ large, may drive the pendulum outside the region. If η_0 is not that large, the pendulum (in the limit) will oscillate around the vertical position indefinitely. A rigorous statement can be derived following standard arguments. We state it now for further reference.

Consider the, standard, Liapunov function

$$H(\theta, \eta) = \int_0^\theta (\hat{V}\ \cos(\rho) - g)\sin(\rho)\mathrm{d}\rho + \frac{1}{2}\eta^2. \tag{5.3}$$

Proposition 5.2. *Given an initial condition (θ_0, η_0), the solution $(\theta(t), \eta(t))$ to (5.1) with this initial condition, satisfies $H(\theta(t), \eta(t)) = \text{constant}$. In particular, if $\hat{V}\ \cos(\theta) > g$ and if (θ_0, η_0) is close to $(0, 0)$, then $(\theta(t), \eta(t))$ stay close to the origin indefinitely.*

Follows easily from the observation that the time derivative of $H(\theta(t), \eta(t))$ is equal to zero.

The structure of the Liapunov function (5.3) reveals the nature of the oscillations of the pendulum. When θ is small, the level curves look like ellipses, with major axis elongated, as $\hat{V}$ gets larger, in the velocity direction. The stronger the vibrations are, the larger the velocities are, while θ vibrates around the vertical position. Notice that in the demonstrations of the Stephenson–Kapitza stability, available on the internet, what we see and sense is the position of the pendulum, and not so much its speed.

It should be pointed out that while the vibrations overcome the gravitational force, they do not make the pendulum asymptotically stable, namely, $\theta(t)$ is not converging to the vertical direction. Some demonstrations on the internet show convergence to the vertical position, and claim that this is the result of the vibrations. It is a mistake. Such a convergence is a result of some friction in the system. See example 6.2.

As for the equation (2.1) when ε is small, namely when the real world equation is an approximation of the limit, the results imply that the smaller the ε, the longer the solution will follow the limit solution. But stability over an infinite time is not guaranteed. In the presence of friction, however, asymptotic stability is maintained.

6. Examples

We collect in this section several variations of the stabilization phenomenon, and some comments on the interpretation of the results.

In many applications, the term $g(x, \frac{dx}{dt})$ is affine in the derivative. The expression of the limit equation is then simpler, as follows,

Corollary 6.1. *If in* (4.1) *the term* $g(x, \frac{dx}{dt})$ *has the form* $g(x) + r(x)\frac{dx}{dt}$, *then the limit equation* (4.2) *takes the form*

$$\frac{d^2x}{dt^2} = g(x) + r(x)\frac{dx}{dt} - \hat{V}f'(x)f(x). \tag{6.1}$$

Follows from (4.2), since the average of $v(\cdot)$ is 0.

Example 6.2. Consider a linear spring with, linear damping and restoring (positive or negative) forces, and with mass subject to violent vibrations satisfying assumption 2.1. The differential equation governing the dynamics is

$$\frac{d^2x}{dt^2} = kx + r\frac{dx}{dt} + \frac{1}{\varepsilon}a\left(\frac{t}{\varepsilon}\right)x. \tag{6.2}$$

The previous considerations imply that solutions to (6.2) converge to solutions of

$$\frac{d^2x}{dt^2} = r\frac{dx}{dt} - (\hat{V} - k)x. \tag{6.3}$$

Therefore, while the violent vibrations may stabilize the spring (likewise, the pendulum) by overcoming the positive restoring force, they do not affect the damping forces. If the damping coefficient r is negative, the spring will converge to the equilibrium. If r is positive, the equilibrium is unstable, regardless how violent the vibrations are. This agrees with the proposed intuition. Indeed, while affecting the position x of the spring, the vibrational forces increase the velocity, thus cannot overcome a force that grows with the velocity.

Example 6.3. We wish to demonstrate the role of the nonlinear dependence on the velocity allowed in the general result. To that end, consider a vibrating linear spring with nonlinear friction, given by

$$\frac{d^2x}{dt^2} = -\left(\frac{dx}{dt}\right)^3 + \frac{1}{\varepsilon}a\left(\frac{t}{\varepsilon}\right)x. \tag{6.4}$$

The cubic term friction here is made up for the purpose of the illustration, but it is not that far from some nonlinear friction terms that appear in the literature. We assume that the acceleration

$a(\cdot)$ satisfies assumption 2.1. As before, we denote $\hat{V}= \int_0^1 v(\sigma)^2 \mathrm{d}\sigma$. Also denote, for the sake of this example, $\hat{W}= \int_0^1 v(\sigma)^3 \mathrm{d}\sigma$. Applying theorem 4.1, and carrying out the integration (and recalling that the average of $v(\cdot)$ is 0), we get that in a region where solutions are uniformly bounded, the solutions to (6.4) converge as $\varepsilon \to 0$, to the solutions to

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\left(\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^3 + 3\hat{V}\frac{\mathrm{d}x}{\mathrm{d}t}x^2 + \hat{W}x^3\right) - \hat{V}x. \tag{6.5}$$

We see, indeed, that due to the nonlinearity of the friction term, the velocities within the vibrations affect the friction of the global dynamics.

Example 6.4. We consider an inverted pendulum with tilted vibrations. Namely the vibrations of the pivot are in a direction of θ_0 degrees from the vertical direction. (Videos of such actions can be found on the internet). Figure 4 depicts the relevant forces. For convenience we choose as the origin $\theta = 0$ the tilted direction.

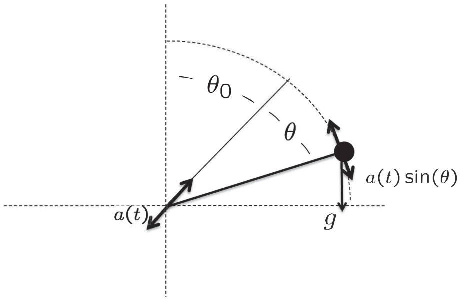

Figure 4.

By applying an acceleration a to the suspension point, the effective acceleration is $a\sin(\theta)$, as in the inverted pendulum. The effective gravitational force is $g\sin(\theta + \theta_0)$. The differential equation that governs the pendulum with tilted vibrations is then

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = g\ \sin(\theta + \theta_0) + \frac{1}{\varepsilon}a\left(\frac{t}{\varepsilon}\right)\sin(\theta). \tag{6.6}$$

Notice that the fast equation of (6.6) is the same as the one related to the vertical pendulum. Likewise, the analysis concerning the limit of the fast contribution in section 4 is not affected by the change of the slow term. Hence in the limit as $\varepsilon \to 0$, solutions of (6.6) satisfy the differential equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = g\ \sin(\theta + \theta_0) - \hat{V}\ \cos(\theta)\sin(\theta). \tag{6.7}$$

The origin (i.e. the direction to which the vibrations are pointing) is not an equilibrium point of the system. The equilibrium is the angle θ_{eq} that solves the equation

$$g\ \sin(\theta_{\mathrm{eq}} + \theta_0) = \hat{V}\ \cos(\theta_{\mathrm{eq}})\sin(\theta_{\mathrm{eq}}). \tag{6.8}$$

This equilibrium is then stable. A Liapunov function around the equilibrium is easy to find.

There are numerous videos on the internet, which demonstrate the preceding example, in particular, a pendulum that, under vibrations, stays in an horizontal position. The analysis also suggests a way to lift a pendulum from the stable position pointing down, to a stable vertical position, using only autonomous vibrations. Indeed, set the vibrations to point in a wide angle, say 160°, from the upward vertical position. With enough vibrational force the pendulum will be lifted toward the 160° direction. Then change slowly, the direction of the vibrations toward the vertical direction. The pendulum will follow, until the stable inverted pendulum position is reached. There are videos on the internet that validate this mathematical prediction.

7. Further models

We display in this section two examples that deviate from the model examined so far in this paper, but that can be analyzed using the same tools.

Example 7.1. As was mentioned in the introduction, the stability of the inverted pendulum was verified also for vibrations of the form $\frac{1}{\varepsilon^2}a\left(\frac{1}{\varepsilon}\right)$. This was done, primarily, using the averaging method. The intuition and the mathematics suggested in the present paper contribute to the understanding of such mechanism. We restrict the discussion to the inverted pendulum. Other systems can be treated in a similar way. The relevant differential equation is

$$\frac{d^2\theta}{dt^2} = \left(g + \frac{1}{\varepsilon^2}a\left(\frac{t}{\varepsilon}\right)\right)\sin(\theta). \tag{7.1}$$

We state it upfront: the limit of solutions to (7.1) as $\varepsilon \to 0$, cannot be described as a solution of an ordinary differential equation. Stability consequences, however, and some properties of solutions near the limit, can be derived.

The intuition displayed in section 2 applies. When θ is positive and $\sin(\theta)$ is strictly increasing, the effective deceleration is more efficient than the effective acceleration, hence yielding vibrational forces toward the upright position. Furthermore, the parameter $\frac{1}{\varepsilon^2}$ in (7.1) generates vibrational forces stronger than those induced by the parameter in (2.1).

A rigorous consequence can be easily obtained under a slight change of the statement. Rather than $\frac{1}{\varepsilon^2}$, consider in (7.1) a split of the parameter in the form $\frac{1}{\varepsilon\delta}$, and ask for the limit as $\varepsilon \to 0$, then $\delta \to 0$. For δ fixed, the analysis in sections 4 and 5 are valid, and the limit as $\varepsilon \to 0$ of the solutions is governed by the equation

$$\frac{d^2\theta}{dt^2} = \left(g - \frac{1}{\delta^2}\hat{V}\ \cos(\theta)\right)\sin(\theta). \tag{7.2}$$

For any $\hat{V}$, even small, if δ is small, the equilibrium is stable. However, the vibrational forces then are enormous, tending to infinity as $\delta \to 0$. The limit as $\delta \to 0$ of solutions to (7.2) is not governed by a differential equation. For δ small we can still consider the relevant Liapunov function, namely (compare with (5.3)),

$$H(\theta, \eta) = \int_0^\theta \left(\frac{1}{\delta^2}\hat{V}\ \cos(\rho) - g\right)\sin(\rho)d\rho + \frac{1}{2}\eta^2. \tag{7.3}$$

It validates the stability, and also showing that a small change in the position may generate an enormous change in the velocity. In the limit as $\delta \to 0$, the resulting velocity becomes infinite (recall that in the standard experiments we sense the movement of the position of the pendulum, but can hardly estimate its velocity).

The same arguments hold when the violent vibrations in (7.1) are replaced by the term $\frac{1}{\varepsilon^\alpha}$, with $\alpha > 1$. Indeed, we can then express the parameter as $\frac{1}{\varepsilon\delta}$, where now δ stands for $\varepsilon^{\alpha-1}$. The rest of the arguments stand.

The observation just made explains why it is relatively easy to demonstrate the stabilization phenomenon. Indeed, one need not produce violent vibrations exactly in the form of (2.1), as any ε^α with $\alpha \geqslant 1$ will do.

If the coefficient in (7.1) is $\frac{1}{\varepsilon^\alpha}$ with $\alpha < 1$, the same argument is still valid, however, with $\frac{1}{\varepsilon^{\alpha-1}}$ converging to 0 as $\varepsilon \to 0$. This means that in the presence of gravitation, the vibrational force, in the limit, will not stabilize the system.

As was mentioned before, the tools suggested in section 3 can be applied to frameworks more general than what we have discussed so far. A variety of applications of the model recalled in section 3 were worked out, providing, both, theoretical and numerical results. We demonstrate now such a possibility, in the context of the inverted pendulum.

Example 7.2. Consider the inverted pendulum example (2.1), but rather than having autonomous vibrations, let the acceleration be of the form $a(t, \eta, \theta)$, namely, a feedback of θ and of $\eta = \frac{d\theta}{dt}$. Let the equation

$$\frac{d^2\theta}{dt^2} = \left(g + \frac{1}{\varepsilon} a\left(\frac{t}{\varepsilon}, \eta, \theta\right)\right) \sin(\theta) \tag{7.4}$$

be arranged such that any solution of the differential equation

$$\frac{d\eta}{ds} = a(s, \eta, \theta)\sin(\theta), \tag{7.5}$$

satisfies

$$|\eta(s) - (\sin(s) + \Delta\ \sin(\theta))| \to 0 \quad \text{as } s \to \infty. \tag{7.6}$$

Standard tracking procedures in control theory can be used to come up with such a feedback. Following the derivations of section 4, the fast system is

$$\begin{aligned} \frac{d\theta}{ds} &= 0 \\ \frac{d\eta}{ds} &= a(s, \eta, \theta)\sin(\theta). \end{aligned} \tag{7.7}$$

The feedback vibrations are arranged such that, given initial conditions (θ, η), there is a unique invariant measure of the fast dynamics, given by the constant θ, and the image in the η-space of the mapping that assigns with $s \in [0, 2\pi]$ the value $\sin(s) + \Delta\sin(\theta)$, with the prescribed parameter Δ.

As in the original pendulum example, the position θ is an orthogonal observable. The differential equation for the observable, resulting from (3.4), is now well defined, and given by

$$\frac{d\theta}{dt} = -\Delta\ \sin(\theta). \tag{7.8}$$

Thus, in the limit, the position θ satisfies a first order differential equation, and if $\Delta > 0$, the position 0 is stable.

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