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9

EXTREMAL SETS OF SUBSETS SATISFYING CONDITIONS INDUCED BY A GRAPH

David E. Daykin and Peter Frankl

ABSTRACT Let $\mathscr{F}$ always denote a set of subsets of $N=\{1,2,\ldots,n\}$. If $\mathscr{F}_1, \mathscr{F}_2$ are such that $X\in\mathscr{F}_1$, $Y\in\mathscr{F}_2$ imply $X\cap Y\neq\varnothing$ and $X\neq Y$, then $2^{-n}\min\{|\mathscr{F}_1|,|\mathscr{F}_2|\}\leq(3-\sqrt{5})/2$, and this is essentially best possible. Now we introduce more general results.

Four conditions which $X, Y\subset N$ might satisfy are I: $X\cap Y\neq\varnothing$, U: $X\cup Y\neq N$, C: if $X\neq Y$ then $X\not\subset Y$, and $\neq$: $X\neq Y$. Let J be a subset of {I, U, C, $\neq$}. Let Γ be a finite loopless graph with vertex set $\{1,2,\ldots,v\}$ and edge set E. Define

$$\lambda(n)=2^{-n}\max\{|\mathscr{F}_1|+\ldots+|\mathscr{F}_v|\}\leq v,$$

$$\mu(n)=2^{-n}\max\{\min\{|\mathscr{F}_1|,\ldots,|\mathscr{F}_v|\}\}\leq 1,$$

where the maxima are over all $\mathscr{F}_1,\ldots,\mathscr{F}_v$ such that $(i,j)\in E$, $X\in\mathscr{F}_i$, $Y\in F_j$ imply X, Y satisfy all the conditions in J.

We study the sequences $\lambda(n)$, $\mu(n)$ and show they have limits λ, μ. They are non-decreasing except possibly if J is C, IC, UC, IUC. Clearly λ, μ depend only on Γ, J.

We determine λ in terms of fractional stability numbers of Γ. The case $J=\mathrm{C}$ for λ generalizes to Kleitman–Lubell–Yamamoto–Meshalkin (or Lubell–Yamamoto–Meshalkin) posets.

When Γ is an edge, $\mu\in\{\frac{1}{4},(3-\sqrt{5})/2,\frac{1}{2}\}$. For Γ arbitrary $\mu(n)=\frac{1}{2}$ for $J=\mathrm{I},\mathrm{U}$ and $\mu(n)=\frac{1}{4}$ for $J=\mathrm{IU}$. When Γ is a directed circuit $\mu(n)=\frac{1}{4}$ for $J=\mathrm{C}{\neq}$. When Γ is undirected and $J=\neq$ we determine μ in terms of the fractional chromatic number of Γ

The paper contains much more information.

1. Introduction

Let n be a positive integer, let N be the set $\{1,2,\ldots,n\}$ and let 2^n be the set of subsets of N. We reserve the letters $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}$ for subsets of 2^n. Four conditions which $X, Y\subset N$ might satisfy are

I	(non-empty intersection)	$X\cap Y\neq\varnothing$;
U	(union not N)	$X\cup Y\neq N$;
C	(not properly contained)	if $X\neq Y$ then $X\not\subset Y$;
$\neq$	(not equal)	$X\neq Y$.

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We reserve the letter J for a subset of {I, U, C, ≠}. For example if $J=\mathrm{C}$ and $\mathcal{F}$ is such that all $X, Y \in \mathcal{F}$ satisfy J, this means that no member of $\mathcal{F}$ is a proper subset of another, in other words that $\mathcal{F}$ is an antichain.

The origins of our work might be traced to a classical result of Sperner. In 1928 he showed that if $\mathcal{F}$ is an antichain then $|\mathcal{F}| \leqslant \mathrm{Sper}(n)$, where

$$\mathrm{Sper}(n)=\binom{n}{\lfloor \frac{1}{2}n \rfloor}=\max_{0 \leqslant i \leqslant n}\left\{\binom{n}{i}\right\}.$$

This result was generalized independently by Yamamoto (in 1954), Meshalkin (in 1963) and Lubell (in 1966) to the well-known Lubell–Yamamoto–Meshalkin (LYM) inequality:

$$\sum_{X \in \mathcal{F}} 1\Big/\binom{n}{|X|} \leqslant 1 \quad \text{if} \quad \mathcal{F} \text{ is an antichain.}$$

In 1972 Brace and Daykin [**3**] studied $\max|\mathcal{F}|$ over all $\mathcal{F}$ such that all $X, Y \in \mathcal{F}$ satisfy a subset J of the conditions I, U, C. They conjectured that $\max|\mathcal{F}|=2^n/4$ when $J=\mathrm{IU}$. This was proved by Daykin, Hilton, Lovász, Schönheim, Seymour and others.

The above results all concern one set $\mathcal{F}$, and further details can be found in the recent survey by West [**15**]. In this paper we have two sets $\mathcal{F}$, $\mathcal{G}$ or more. Just one of our results is that $\max\{\min\{|\mathcal{F}|, |\mathcal{G}|\}\}=2^n/4$ over all $\mathcal{F}$, $\mathcal{G}$ such that all $X \in \mathcal{F}$, $Y \in \mathcal{G}$ satisfy IU. This is clearly stronger than the above result about IU.

2. Statement of main results

Throughout this paper Γ will be a finite loopless graph without multiple edges having vertex set $\{1, 2, \ldots, v\}$ and edge set $E(\Gamma)$. We assume that $E(\Gamma) \neq \varnothing$ so $v \geqslant 2$. We think of sets $\mathcal{F}_1, \ldots, \mathcal{F}_v$ as sitting at the vertices of Γ.

Let J be a non-empty subset of the conditions I, U, C, ≠. Define

$$\lambda(n)=2^{-n}\max\{|\mathcal{F}_1|+\ldots+|\mathcal{F}_v|\} \leqslant v,$$

$$\mu(n)=2^{-n}\max\{\min\{|\mathcal{F}_1|, \ldots, |\mathcal{F}_v|\}\} \leqslant 1,$$

where these maxima are over all $\mathcal{F}_1, \ldots, \mathcal{F}_v$ such that $(i, j) \in E(\Gamma)$, $X \in \mathcal{F}_i$, $Y \in \mathcal{F}_j$ imply that X, Y satisfy all the conditions in J.

The motive for this paper was to study the sequences $\lambda(1), \lambda(2), \ldots$ and $\mu(1), \mu(2), \ldots$. They both depend on Γ and J and they are non-decreasing except possibly if J is C, IC, UC or IUC. We show in Section 3

that they always have limits λ, μ respectively. Clearly λ, μ depend only on Γ and J. Now Γ may be directed or undirected but if $\mathrm{C} \notin J$ this makes no difference. So when Γ is directed we always assume $\mathrm{C} \in J$.

Results on $\lambda(n)$ and λ

These are virtually complete. We remind the reader that a subset A of the vertex set $\{1, \ldots, v\}$ is independent if $i, j \in A$ implies that $(i, j) \notin E(\Gamma)$. Also the independence number $\alpha(\Gamma)$ of Γ is $\max\{|A|\}$ over independent sets A. Suppose that A is independent and $|A|=\alpha(\Gamma)$. For $1 \leqslant i \leqslant v$ put $\mathcal{F}_i=2^n$ if $i \in A$ but $\mathcal{F}_i=\varnothing$ if $i \notin A$. This example is IUC≠ and shows that $\alpha(\Gamma) \leqslant \lambda(n)$ always. In Section 4 we will define $\alpha^{(1)}(\Gamma)$ and $\alpha^{(2)}(\Gamma)$ for

THEOREM 2.1 *If Γ is undirected then*

$$\lambda(n)=\begin{cases}\alpha^{(1)}(\Gamma) & \text{for } J=\mathrm{I}, \mathrm{U},\\ \alpha^{(2)}(\Gamma) & \text{if } J=\mathrm{IU},\\ \alpha(\Gamma) & \text{if } \neq \in J,\\ \alpha(\Gamma) & \text{for } n \geqslant n_0=n_0(\Gamma) \text{ and } J=\mathrm{C}, \mathrm{IC}, \mathrm{UC}, \mathrm{IUC}.\end{cases}$$

When Γ is a triangle n_0 is 5, 2, 2, 1 as J is C, IC, UC, IUC.

THEOREM 2.2 *If Γ is directed then $\lambda=\alpha(\Gamma)$ if $\mathrm{C} \in J$ and $\lambda(n)=\alpha(\Gamma)$ if C, ≠ ∈ J.*

THEOREM 2.3 *If Γ is a directed edge and $\mathrm{C} \in J$ then $\alpha(\Gamma)=1$ and*

$$\lambda(n)=\begin{cases}1+2^{-n}\,\mathrm{Sper}(n) & \text{if } J=\mathrm{C},\\ 1 & \text{otherwise.}\end{cases}$$

In view of these three theorems it only remains to evaluate $\lambda(n)$ for directed Γ in the cases C, IC, UC, IUC.

Results on $\mu(n)$ and μ

First we have a lower bound for $\mu(n)$.

LEMMA 2.1 *For $m=2, 3, \ldots$ we have*

$$2^{1-2m} \leqslant \mu(n) \text{ for all } J, \Gamma \text{ with } v \leqslant \binom{2m-1}{m} \text{ and } 2m \leqslant n.$$

A pleasing discovery is

THEOREM 2.4

$$\mu(n)=\begin{cases}\frac{1}{2} & \text{for } J=\mathrm{I}, \mathrm{U},\\ \frac{1}{4} & \text{if } J=\mathrm{IU}.\end{cases}$$

We get an upper bound for $\mu(n)$ by looking at a single edge of Γ. When Γ is undirected and bipartite, doing so is in fact sufficient to evaluate $\mu(n)$. It turns out that when Γ is an edge the only possible values of μ are $\frac{1}{4}$, $(3-\sqrt{5})/2=0.381\,97$ and $\frac{1}{2}$, as shown in Table 2.1.

TABLE 2.1 Values of μ when Γ is an edge

Undirected	Directed	μ
I, U, $\neq$	C, C$\neq$	$\frac{1}{2}$
I$\neq$, U$\neq$	IC, UC, IC$\neq$, UC$\neq$	$(3-\sqrt{5})/2$
Otherwise	IUC, IUC$\neq$	$\frac{1}{4}$

When Γ is a directed circuit we prove that $\mu(n)=\frac{1}{4}$ if J is C$\neq$ in Theorem 11.1. When Γ is undirected and J is $\neq$ we determine μ in terms of the fractional chromatic number of Γ in Theorem 10.1. Clearly there is much remaining to be discovered about $\mu(n)$.

Results for Kleitman–LYM (KLYM) posets

Let P be a KLYM poset. The definition is given in Section 5, and 2^n is a KLYM poset. These seem to be the natural setting for dealing with the case $J=\mathrm{C}$ of 2^n. For $1\leqslant i\leqslant v$ let F_i be a subset of the elements of P. In this context the condition C takes the form

$$(i,j)\in E(\Gamma),\qquad p\in F_i,\qquad q\in F_j,\qquad p\neq q\quad\text{imply}\quad p\not<q. \tag{2.1}$$

Let m be the maximum of the cardinalities of the ranks of P. We have three results in Section 5.

THEOREM 2.5 *If Γ is undirected and* (2.1) *holds and*

$$m(v+1-\alpha(\Gamma))\alpha(\Gamma)\leqslant|P|, \tag{2.2}$$

then

$$|P|^{-1}\max\{\textstyle\sum|F_i|\}=\alpha(\Gamma).$$

THEOREM 2.6 *If Γ is directed and* (2.1) *holds then*

$$\alpha(\Gamma)\leqslant|P|^{-1}\max\{\textstyle\sum|F_i|\}\leqslant\alpha(\Gamma)+|P|^{-1}m(v-1),$$

and these bounds are best possible.

THEOREM 2.7 *If Γ is directed and* (2.1) *holds and condition $\neq$ holds then*

$$|P|^{-1}\max\{\textstyle\sum|F_i|\}=\alpha(\Gamma).$$

3. The convergence of $\lambda(n)$ and $\mu(n)$

Let $\mathscr{F}_1,\ldots,\mathscr{F}_v$ be an example for $n=m\geqslant 2$ with properties J. For $1\leqslant i\leqslant v$ put

$$\mathscr{G}_i=\{X\setminus\{m\}: m\in X\in F_i\}\quad\text{and}\quad \mathscr{H}_i=\{X: m\notin X\in\mathscr{F}_i\}.$$

Then $\mathscr{G}_1,\ldots,\mathscr{G}_v$ has properties $J\setminus\mathrm{I}$ and $\mathscr{H}_1,\ldots,\mathscr{H}_v$ has properties $J\setminus\mathrm{U}$ for $n=m-1$. This proves that

$$\max\{\lambda_{J\setminus\mathrm{I}}(n),\lambda_{J\setminus\mathrm{U}}(n)\}\geqslant\lambda_J(n+1)\quad\text{for}\quad n=1,2,\ldots$$

and in particular establishes

LEMMA 3.1 *We have $\lambda(1)\geqslant\lambda(2)\geqslant\ldots\geqslant 1$ for $J=\mathrm{C},\neq,\mathrm{C}\neq$.* □

Suppose we have an example $\mathscr{F}_1,\ldots,\mathscr{F}_v$ for $n=m-1\geqslant 1$. We form a second example for $n=m$ by replacing each member X of each $\mathscr{F}_i$ by the two sets X and $X\cup\{m\}$. We call this operation *doubling*. The properties I, U, $\neq$, C$\neq$ clearly carry over under doubling, and this fact yields

LEMMA 3.2 *We have $\lambda(1)\leqslant\lambda(2)\leqslant\ldots\leqslant v$ and $\mu(1)\leqslant\mu(2)\leqslant\ldots\leqslant 1$ except possibly if J is* C, IC, UC *or* IUC. □

Next suppose that $\mathscr{F}_1,\ldots,\mathscr{F}_v$ is an example with properties J and C$\in J$. For $1\leqslant i\leqslant v$ put

$$\mathscr{G}_i=\mathscr{F}_i\setminus\mathscr{H}_i\quad\text{where}\quad\mathscr{H}_i=\mathscr{F}_i\cap\Big(\bigcup_{(i,j)\in E(\Gamma)}\mathscr{F}_j\Big).$$

Then $\mathscr{H}_i$ is an antichain, and so

$$|\mathscr{F}_i|-\mathrm{Sper}(n)\leqslant|\mathscr{F}_i|-|\mathscr{H}_i|\leqslant|\mathscr{G}_i|.$$

Also $\mathscr{G}_1,\ldots,\mathscr{G}_v$ has properties J and $\neq$. For any $k>n$ let us double $\mathscr{G}_1,\ldots,\mathscr{G}_v$ a total of $k-n$ times. The resulting example has properties J. This shows that

$$\lambda(n)-v2^{-n}\,\mathrm{Sper}(n)\leqslant\lambda(k)\quad\text{for}\quad n\leqslant k \text{ if } \mathrm{C}\in J,$$

and similarly for $\mu(n)$, except that the factor v is deleted. It is now easy to see that for all J the sequences $\lambda(n)$ and $\mu(n)$ converge with limits λ and μ respectively because $2^{-n}\,\mathrm{Sper}(n)\to 0$ as $n\to\infty$.

4. Independence numbers of Γ

A subset A of $V(\Gamma)$ is independent if $i, j \in A$ implies $(i,j)\notin E(\Gamma)$. The independence number $\alpha(\Gamma)$ of Γ is $\max\{|A|\}$ over independent sets A. If $r\geq 1$ the rth power fractional independence number $\alpha^{(r)}(\Gamma)$ of Γ is

$$\max\{x_1^r+\ldots+x_v^r\}$$

over all choices of real numbers $x_1,\ldots,x_v$ such that

$$0\leq x_i\leq 1 \quad \text{for} \quad 1\leq i\leq v, \tag{4.1}$$

$$x_i+x_j\leq 1 \quad \text{for all} \quad (i,j)\in E(\Gamma). \tag{4.2}$$

Clearly $\alpha^{(r)}(\Gamma)$ decreases as r increases and is not less than $\alpha(\Gamma)$. We need

LEMMA 4.1 *We can achieve $\alpha^{(r)}(\Gamma)$ with all $x_i\in\{0,\frac{1}{2},1\}$, and when $r>1$ we cannot achieve it in any other way.*

PROOF Suppose some $x_i\notin\{0,\frac{1}{2},1\}$. Put $\alpha=\sum x_k^r$,

$$a=\min\{x_i: 0<x_i<1\}, \qquad A=\{i: 1\leq i\leq v, x_i=a\},$$

$$b=\max\{x_i: 0<x_i<1\}, \qquad B=\{i: 1\leq i\leq v, x_i=b\},$$

so $A\neq\varnothing$, $B\neq\varnothing$ and $a\leq b$.

Case $a+b<1$ Here if $i\in A$ and $(i,j)\in E(\Gamma)$ then $x_i+x_j<1$ so we can increase x_i and hence α.

Case $a+b>1$ Here if $j\in B$ and $(i,j)\in E(\Gamma)$ then $x_i=0$ so we can increase x_j.

Case $a+b=1$ Here we eliminate $b=1-a$ and consider α as a function of a, keeping all x_i with $i\notin A\cup B$ constant. The result follows by the convexity of α. □

5. Theorems on KLYM posets

Let P be a ranked poset. That means there is a map rank: $P\to\{0,1,\ldots,h\}$ such that for all $p,q\in P$ with $p<q$, firstly $\mathrm{rank}(p)<\mathrm{rank}(q)$, and secondly if $2+\mathrm{rank}(p)\leq\mathrm{rank}(q)$ then $p<p'<q$ for some $p'\in P$. We assume that for $0\leq i\leq h$ rank i of P, which is the set $\{p\in P: \mathrm{rank}(p)=i\}$, has cardinality $\nu(i)\geq 1$. We put $m=\max\{\nu(0),\ldots,\nu(h)\}$ and clearly $\nu(0)+\ldots+\nu(h)=|P|$.

Suppose further that P is a KLYM poset. By this we mean that there is non-empty list $\Lambda_1,\ldots,\Lambda_c$ of maximal chains Λ_i of P such that for every $p\in P$ the number of chains which contain p in the list is $c/\nu(\mathrm{rank}(p))$. There is a growing literature about KLYM posets, cf [**1**, **2**, **4**, **6–8**, **10**, **13**, **15**]. Other writers have called them LYM posets, but our K is in honour of Kleitman [**13**]. The set 2^n of subsets of N is perhaps the most important example of a KLYM poset.

For $1\leq i\leq v$ let F_i be a subset of the elements of P. We are here interested in

$$|P|^{-1}\max\{|F_1|+\ldots+|F_v|\}$$

over all $F_1,\ldots,F_v$ satisfying conditions like I, U, C, $\neq$. However, conditions I, U do not carry over to KLYM posets. Further, the maximum is trivially $|P|^{-1}\,\alpha(\Gamma)$ if $\neq$ is among the conditions. Hence we are only left with condition C, which here takes the form (2.1). Our results are Theorems 2.5, 2.6 and 2.7.

PROOF OF THEOREM 2.5 Let $F_1,\ldots,F_v$ satisfying (2.1) be given. For any maximal chain Λ of P consider a bipartite graph $\Delta=\Delta(\Lambda)$. If Λ is $p_0<p_1<\ldots<p_h$ say, then the vertices in one part of Δ are $p_0,p_1,\ldots,p_h$ and the vertices in the other part are $F_1,F_2,\ldots,F_v$. We have (p_i,F_j) as an edge of Δ if and only if $p_i\in F_j$.

Suppose that Δ had $\alpha+1$ independent edges $(P_{i_0},F_{j_0}),\ldots,(P_{i_\alpha},F_{j_\alpha})$. Then by definition of α there are $j_r\neq j_s$ such that $(j_r,j_s)\in E(\Gamma)$. Since $p_{i_r}\in F_{j_r}$ and $p_{i_s}\in F_{j_s}$ and either $p_{i_r}<p_{i_s}$ or $p_{i_s}<p_{i_r}$ this contradicts (2.1). Hence Δ does not have $\alpha+1$ independent edges.

We now distinguish two types of maximal chain Λ.

Type 1 $\deg(p \text{ in } \Delta(\Lambda))\leq\alpha$ for all $p\in\Lambda$.

Type 2 Not type 1. For this type of Λ there is some $p\in\Lambda$ with $\deg(p$ in $\Delta(\Lambda))\geq\alpha+1$. Put

$$\theta(\Lambda)=\{p\in\Lambda, \deg(p \text{ in } \Delta(\Lambda))\geq\alpha\}.$$

Since Δ does not have $\alpha+1$ independent edges, by Hall's theorem on systems of distinct representatives, we know that $|\theta(\Lambda)|\leq\alpha$.

Next we claim that for any Λ we have

$$\sum_{p\in\Lambda}\nu(\mathrm{rank}(p))\deg(p \text{ in } \Delta(\Lambda))\leq\alpha\,|P|. \tag{5.1}$$

For chains of Type 1 this is trivial. For chains of Type 2 the left side of (5.1) is not greater than

$$\begin{aligned}&(\alpha-1)\sum_{\substack{p\in\Lambda\\ \deg p\le\alpha-1}}\nu(\text{rank}(p))+v\sum_{\substack{p\in\Lambda\\ \deg p\ge\alpha}}\nu(\text{rank}(p))\\ &=(\alpha-1)\sum_{p\in\Lambda}\nu(\text{rank}(p))+(v-\alpha+1)\sum_{\substack{p\in\Lambda\\ \deg p\ge\alpha}}\nu(\text{rank}(p))\\ &\le(\alpha-1)|P|+(v-\alpha+1)\sum_{\substack{p\in\Lambda\\ \deg p\ge\alpha}}m\\ &\le(\alpha-1)|P|+(v-\alpha+1)\alpha m\\ &\le\alpha|P|,\end{aligned}$$

by (2.2). So (5.1) holds for any Λ.

Next let χ be the characteristic function defined by $\chi(w, W)$ is 1 if $w\in W$ but 0 otherwise. Then clearly we have

$$\deg(p_i \text{ in } \Delta(\Lambda))=\sum_{1\le j\le v}\chi(p_i, F_j).$$

Also for any $F\subset P$ we have

$$c\,|F|=\sum_{\Lambda\in\text{list}}\sum_{p\in\Lambda}\nu(\text{rank}(p))\chi(p, F).$$

It now follows that

$$\begin{aligned}c\sum_{1\le j\le v}|F_j|&=\sum_{\Lambda\in\text{list}}\sum_{p\in\Lambda}\nu(\text{rank}(p))\sum_{1\le j\le v}\chi(p, F_j)\\ &=\sum_{\Lambda\in\text{list}}\sum_{p\in\Lambda}\nu(\text{rank}(p))\deg(p \text{ in } \Delta(\Lambda))\\ &\le\sum_{\Lambda\in\text{list}}\alpha\,|P|=c\alpha\,|P|,\end{aligned}\qquad(5.2)$$

and Theorem 2.5 is proved. □

PROOF OF THEOREM 2.6 Let $F_1, \ldots, F_v$ satisfying (2.1) be given. For any maximal chain Λ let $\Delta(\Lambda)$ be as defined in the last proof. Since Γ is now directed we modify $\Delta(\Lambda)$ to get a bipartite graph $\Delta'(\Lambda)$. For $1\le j\le v$, if there is an edge (p_i, F_j) in Δ, we remove the one with i as small as possible from Δ. Thus if R is the set of removed edges then $|R|\le v$. Finally, if $R\ne\varnothing$ we choose exactly one edge $(p_i, F_j)\in R$ with i as small as possible and replace it. The result is Δ'. Notice that Δ' is a subgraph of Δ obtained by deleting at most $v-1$ edges.

We claim that $\deg(p_i \text{ in } \Delta'(\Lambda))\le\alpha$ for all Λ and $0\le i\le h$. If true we can bound the sum in (5.1) as follows:

$$\begin{aligned}\sum_{p\in\Lambda}\nu(\text{rank}(p))\deg(p \text{ in } \Delta)&\le m(v-1)+\sum_{p\in\Lambda}\nu(\text{rank}(p))\deg(p \text{ in } \Delta')\\ &\le m(v-1)+\alpha\sum_{p\in\Lambda}\nu(\text{rank}(p))\\ &=m(v-1)+\alpha\,|P|.\end{aligned}\qquad(5.3)$$

The term $m(v-1)$ above arises because an edge (p, F) in $\Delta\setminus\Delta'$ contributes $\nu(\text{rank}(p))\le m$ to the sum, and there are not more than $v-1$ such edges. Using (5.3) in (5.2) instead of (5.1) we get the right-hand side inequality of Theorem 2.6. The left-hand side is trivial, so it remains to establish our claim that $\deg(p \text{ in } \Delta')\le\alpha$.

Suppose therefore that there is a k with $\deg(p_k \text{ in } \Delta')\ge\alpha+1$. Then by definition of α there are r, s such that (p_k, F_r), $(p_k, F_s)\in\Delta'$ and $(r, s)\in E(\Gamma)$. In turn, by definition of R, there exist (p_i, F_r), $(p_j, F_s)\in R$. Recall that we replaced one edge, e say, of R. If $e=(p_k, F_r)$ then $k=i$ and $e\ne(p_k, F_s)$ so $j<k$, contradicting the definition of e. Hence $e\ne(p_k, F_r)$ and $i<k$. We now have $(r, s)\in E(\Gamma)$, $p_i\in F_r$, $p_k\in F_s$ and $p_i<p_k$. This contradicts (2.1), establishes our claim, and ends the proof of the inequality of Theorem 2.6.

Finally we give an example to show that the upper bound can be attained. Let Γ be the complete directed transitive graph so $(i, j)\in E(\Gamma)$ if and only if $1\le i<j\le v$. Let P be a KLYM poset. Choose i with $\nu(i)=m$. Put $F_1=\{p\in P\colon \text{rank}(p)\ge i\}$, $F_2=\ldots=F_{v-1}=\text{rank } i$ of P, and $F_v=\{p\in P\colon \text{rank}(p)\le i\}$. Then $\sum|F_i|=|P|+m(v-1)$ as desired. □

PROOF OF THEOREM 2.2 Put $P=2^n$ in Theorems 2.6 and 2.7. □

6. To find $\lambda(n)$ when Γ is undirected

PROOF OF THEOREM 2.1 This proof is constructed for the various cases as follows:

Case $J=\mathrm{I}$ Let $\mathcal{F}_1, \ldots, \mathcal{F}_v$ satisfy $J=\mathrm{I}$. This means that if $(i, j)\in E(\Gamma)$ and $X\in\mathcal{F}_i$ and $Y\in\mathcal{F}_j$ then $X\cap Y\ne\varnothing$. Put $x_i=2^{-n}\,|\mathcal{F}_i|$ for $1\le i\le v$. Then clearly (4.1) holds. Also (4.2) holds, for if $x_i+x_j>1$ there is an $X\subset N$ such that $X\in\mathcal{F}_i$ and $N\setminus X\in\mathcal{F}_j$. Hence $\sum x_i\le\alpha^{(1)}(\Gamma)$ or in other words $2^{-n}(|\mathcal{F}_1|+\ldots+|\mathcal{F}_v|)\le\alpha^{(1)}(\Gamma)$.

To show that this is best possible we must give an example. For use here and elsewhere put

$$\mathscr{G}_{i,j}=\{X\subset N: i\in X, j\notin X\} \quad \text{for} \quad 1\leq i, j\leq n;$$

thus

$$\mathscr{G}_{1,0}=\{X\subset N: 1\in X\} \quad \text{and} \quad \mathscr{G}_{0,1}=\{X\subset N: 1\notin X\}.$$

As shown in Lemma 4.1 we can choose $x_i\in\{0,\frac{1}{2},1\}$ with $\sum x_i=\alpha^{(1)}(\Gamma)$. Then for $1\leq i\leq v$ we let $\mathscr{F}_i$ be $\varnothing$, $\mathscr{G}_{1,0}$, 2^n accordingly as x_i is $0,\frac{1}{2},1$. This example completes the proof of case $J=\mathrm{I}$.

Case $J=\mathrm{U}$ This follows from case $J=\mathrm{I}$ by replacing each set X by its complement $N\backslash X$.

We will need the fundamental result

LEMMA 6.1 (Seymour [14]) $\sqrt{|\mathscr{F}|}+\sqrt{|\text{incomp }\mathscr{F}|}\leq\sqrt{(2^n)}$.

Here and elsewhere

$$\text{incomp }\mathscr{F}=\{Y\subset N: \text{both } X\not\subset Y \text{ and } Y\not\subset X \text{ for all } X\in\mathscr{F}\}.$$

Note that we use $\subset$ in the sense that $X\subset Y$ allows $X=Y$.

Case $J=\mathrm{IU}$ Let $\mathscr{F}_1,\ldots,\mathscr{F}_v$ satisfy $J=\mathrm{IU}$. If $(i,j)\in E(\Gamma)$ then $\mathscr{F}_j^c=\{N\backslash X: X\in\mathscr{F}_j\}\subset\text{incomp }\mathscr{F}_i$ so $\sqrt{|\mathscr{F}_i|}+\sqrt{|\mathscr{F}_j^c|}=\sqrt{|\mathscr{F}_i|}+\sqrt{|\mathscr{F}_j|}\leq\sqrt{2^n}$. We put $x_i=\sqrt{(2^{-n}|\mathscr{F}_i|)}$ for $1\leq i\leq v$. Then (4.1), (4.2) hold so $\sum x_i^2\leq\alpha^{(2)}(\Lambda)$ or $2^{-n}\sum|\mathscr{F}|\leq\alpha^{(2)}(\Gamma)$.

For our example let $\mathscr{F}=\mathscr{G}_{1,2}$, so $\mathscr{F}$ is clearly IU. We choose $x_i\in\{0,\frac{1}{2},1\}$ with $\sum x_i^2=\alpha^{(2)}(\Gamma)$. Then for $1\leq i\leq v$ we let $\mathscr{F}_i$ be $\varnothing,\mathscr{F},2^n$ according as x_i is $0,\frac{1}{2},1$. This example completes the proof of the case $J=\mathrm{IU}$.

Case $\neq\in J$ We always have $\alpha(\Gamma)\leq\lambda(n)$. Given $X\subset N$ let A be the set of vertices i such that $X\in\mathscr{F}_i$. Because $\neq\in J$ this set A is independent so $|A|\leq\alpha(\Gamma)$. It follows immediately that $2^{-n}\sum|\mathscr{F}|\leq\alpha(\Gamma)$. Notice that we did not use the structure of 2^n, so if 2^n was replaced by any set B we would have $\max\sum|\mathscr{F}|=|B|\,\alpha(\Gamma)$.

Cases $J=\mathrm{C}$, IC, UC, IUC These follow from Theorem 2.5 with $P=2^n$ because then (2.2) holds for n sufficiently large. □

7. Two examples

Let a be a real number in $0<a<\frac{1}{2}$. Let $N=\{1,\ldots,n\}$ be partitioned into $N=M_1\cup\ldots\cup M_k$. For $1\leq i\leq k$ let $m_i=|M_i|$ and do the following. Choose any ordering $<_i$ of the set of all subsets of M_i, such that $Z, Z'\subset M_i$ and $|Z|<|Z'|$ imply $Z<_iZ'$. Let $\mathscr{A}_i$ be the first $\lfloor a2^{m_i}\rfloor$ subsets of M_i in this ordering. Let $\mathscr{C}_i$ be complementary to $\mathscr{A}_i$ so $\mathscr{C}_i=\{M_i\backslash Z: Z\in\mathscr{A}_i\}$. Let $\mathscr{B}_i$ be the remaining subsets of M_i so $\mathscr{B}_i=$

$\{Z\subset M_i: Z\notin\mathscr{A}_i\cup\mathscr{C}_i\}$ and $|\mathscr{B}_i|\sim(1-2a)2^{m_i}$. Notice carefully that $\mathscr{A}_i,\mathscr{B}_i,\mathscr{C}_i$ are pairwise disjoint. Moreover if $X\in\mathscr{A}_i$, $Y\in\mathscr{B}_i$ and $Z\in\mathscr{C}_i$ then $Y\cap Z\neq\varnothing$, $X\cup Y\neq N$, $Y\not\subset X$, $Z\not\subset Y$ and $X\neq Y\neq Z$.

Next we define

$$\mathscr{F}^{(1)}=\{X\subset N: X\cap M_i\in\mathscr{C}_i \text{ for at least one } i\},$$
$$\mathscr{F}^{(2)}=\{Z\subset N: Z\cap M_i\in\mathscr{A}_i\cup\mathscr{B}_i \text{ for all } i\},$$
$$\mathscr{G}=\{Y\subset N;\ Y\cap M_i\in\mathscr{B}_i \text{ for all } i\}.$$

Clearly if $X\in\mathscr{F}^{(1)}$ and $Y\in\mathscr{G}$ then $X\cap Y\neq\varnothing$, $X\not\subset Y$ and $X\neq Y$. Using the fact that $m_1+\ldots+m_k=n$ we obtain

$$|\mathscr{G}|=\prod_{1\leq i\leq k}|\mathscr{B}_i|\sim2^n(1-2a)^k=2^n(1-a)^{2k}b,$$

where

$$b=((1-2a)/(1-a)^2)^k<1,$$

and

$$|\mathscr{F}^{(2)}|\sim2^n(1-a)^k,$$

so

$$|\mathscr{F}^{(1)}|=2^n-|\mathscr{F}^{(2)}|\sim2^n(1-(1-a)^k).$$

We choose a so that $(1-a)^{2k}=1-(1-a)^k=c$ say. Then $c=(3-\sqrt{5})/2$ because $(1-a)^k$ is the golden section $(-1+\sqrt{5})/2$. We will use the fact that as $k,m_1,\ldots,m_k$ all go to ∞ we have $a\to0$ and $b\to1$.

With a new value of a and $\mathscr{G}$ as above we put

$$\mathscr{F}^{(3)}=\{X\subset N: \text{both } X\cap M_i\in\mathscr{A}_i \text{ for at least one } i \text{ and } X\cap M_i\in\mathscr{C}_i \text{ for at least one } i\}.$$

Clearly if $X\in\mathscr{F}^{(3)}$ and $Y\in\mathscr{G}$ then $X\cap Y\neq\varnothing$, $X\cup Y\neq N$, $X\not\subset Y$, $Y\not\subset X$ and $X\neq Y$. Now if

$$\mathscr{F}^{(4)}=\{Z\subset N: Z\in\mathscr{B}_i\cup\mathscr{C}_i \text{ for all } i\},$$

then $|\mathscr{F}^{(2)}|=|\mathscr{F}^{(4)}|$ so

$$|\mathscr{F}^{(3)}|=2^n-|\mathscr{F}^{(2)}|-|\mathscr{F}^{(4)}|+|\mathscr{G}|\sim2^n\{1-2(1-a)^k+(1-2a)^k\}.$$

We now choose a so that $(1-a)^k=\frac{1}{2}$ and then $|\mathscr{F}^{(3)}|\to2^n/4$ and $|\mathscr{G}|\to2^n/4$ as $k,m_1,\ldots,m_k$ all $\to\infty$.

EXAMPLE 7.1 Let $v=2$, $\mathscr{F}_1=\mathscr{F}^{(1)}$, $\mathscr{F}_2=\mathscr{G}$ and Γ be the edge $(1,2)$. As $k,m_1,\ldots,m_k$ all go to ∞ we have $2^{-n}\min\{|\mathscr{F}_1|,|\mathscr{F}_2|\}\to(3-\sqrt{5})/2$. If Γ is undirected then I$\neq$ hold but if Γ is directed then IC$\neq$ hold.

EXAMPLE 7.2 Let $v=2$, $\mathscr{F}_1=\mathscr{F}^{(3)}$, $\mathscr{F}_2=\mathscr{G}$ and Γ be the edge $(1,2)$.

Whether Γ is directed or undirected IUC$\neq$ hold and $2^{-n}\min\{|\mathcal{F}_1|, |\mathcal{F}_2|\} \to \frac{1}{4}$.

8. To find $\lambda(n)$, $\mu(n)$ when Γ is an undirected edge

We have $v=2$, $E(\Gamma)=\{(1,2)\}$ and write $\mathcal{F}$ for $\mathcal{F}_1$ and $\mathcal{G}$ for $\mathcal{F}_2$.

THEOREM 8.1 *If Γ is an undirected edge then $\lambda(n)=1$ for all J and $n\geq 1$.*

PROOF We have $\alpha=\alpha^{(1)}=\alpha^{(2)}=1$, and we apply Theorems 2.1 and 2.5. □

THEOREM 8.2 *When Γ is an undirected edge,*

$$\begin{array}{rll} \mu(n)=\frac{1}{2} & \textit{for} & J=\mathrm{I,U},\neq, \\ \mu(n)\leq\mu=(3-\sqrt{5})/2 & \textit{for} & J=\mathrm{I}\neq,\mathrm{U}\neq, \\ \mu(n)\geq\mu=\frac{1}{4} & \textit{if} & J=\mathrm{C}, \\ \mu(n)=\frac{1}{4} & \textit{for} & J=\mathrm{IU},\mathrm{C}\neq, \\ \mu(n)\leq\mu=\frac{1}{4} & \textit{if} & \mathrm{IU}\subset J \textit{ or } \mathrm{C}\neq\subset J, \\ \mu=\frac{1}{4} & \textit{for} & J=\mathrm{IC},\mathrm{UC}. \end{array}$$

REMARK 8.1 The authors conjectured in **[6]** that $\mu(n)=\max\{\frac{1}{4}, 2^{-n}\,\mathrm{Sper}(n)\}$ for $J=\mathrm{C}$. There is an earlier weaker conjecture of Gronau **[9]**.

PROOF OF THEOREM 8.2 This proof is again presented in terms of the various cases:

Cases $J=\mathrm{I,U},\neq$ By Theorem 8.1 we have $|\mathcal{F}|+|\mathcal{G}|\leq 2^n$ so $2^{-n}\min\{|\mathcal{F}|,|\mathcal{G}|\}\leq\frac{1}{2}$ so $\mu(n)\leq\frac{1}{2}$. To see that we have $\mu(n)=\frac{1}{2}$ consider examples where $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{G}_{0,1}$ or $\mathcal{G}_{1,0}$.

Case $J=\mathrm{C}$ The example $\mathcal{F}=\mathcal{G}_{1,2}$ and $\mathcal{G}=\mathcal{G}_{2,1}$ shows that $\frac{1}{4}\leq\mu(n)$. Then the example $\mathcal{F}=\mathcal{G}=\{X\subset N: |X|=\lfloor\frac{1}{2}n\rfloor\}$ shows that $2^{-n}\,\mathrm{Sper}(n)\leq\mu(n)$. Thus we have the left side of our result

$$\max\{\tfrac{1}{4}, 2^{-n}\,\mathrm{Sper}(n)\}\leq\mu(n)\leq\tfrac{1}{4}+2^{-n}\,\mathrm{Sper}(n).$$

For the right-hand inequality let $\mathcal{A}=\mathcal{F}\cap\mathcal{G}$. Then $\mathcal{A}$ is an antichain so $|\mathcal{A}|\leq\mathrm{Sper}(n)$. Put $\mathcal{F}'=\mathcal{F}\setminus\mathcal{A}$ and $\mathcal{G}'=\mathcal{G}\setminus\mathcal{A}$, then $\mathcal{G}'\subset\mathrm{incomp}\,\mathcal{F}'$, so by Lemma 6.1 we have $\sqrt{|\mathcal{F}'|}+\sqrt{|\mathcal{G}'|}\leq\sqrt{2^n}$. Hence $2^{-n}\min\{|\mathcal{F}'|,|\mathcal{G}'|\}\leq\frac{1}{4}$ and the inequality follows.

Case $J=\mathrm{C}\neq$ The example $\mathcal{F}=\mathcal{G}_{1,2}$ and $\mathcal{G}=\mathcal{G}_{2,1}$ shows that $\frac{1}{4}\leq\mu(n)$. The reverse inequality follows from Lemma 6.1 as in the case $J=\mathrm{C}$ because here $\mathcal{F}\subset\mathrm{incomp}\,\mathcal{G}$.

Case $J=\mathrm{IU}$ We use the example $\mathcal{F}=\mathcal{G}=\mathcal{G}_{1,2}$. Then we note that $\mathcal{F}\subset\mathrm{incomp}\{N\setminus X: X\in\mathcal{G}\}$ and apply Lemma 6.1.

Case $\mathrm{IU}\subset J$ By the case $J=\mathrm{IU}$ we have $\mu(n)\leq\frac{1}{4}$. Then Example 7.2 shows that $\frac{1}{4}\leq\mu$.

Case $\mathrm{C}\neq\subset J$ Here, by Lemma 3.2, we have $\mu(1)\leq\mu(2)\leq\ldots\leq\mu$. By the case $J=\mathrm{C}$ we have $\mu\leq\frac{1}{4}$. Then Example 7.2 shows that $\frac{1}{4}\leq\mu$.

Cases $J=\mathrm{IC},\mathrm{UC}$ We have $\mu\leq\frac{1}{4}$ by the case $J=\mathrm{C}$ and $\frac{1}{4}\leq\mu$ by Example 7.2.

Cases $J=\mathrm{I}\neq,\mathrm{U}\neq$ We will only deal with the case $J=\mathrm{I}\neq$ because the case $J=\mathrm{U}\neq$ will then follow by taking complements in N. In view of Example 7.1 it is sufficient to show that $\mu(n)\leq(3-\sqrt{5})/2$ for $J=\mathrm{I}\neq$.

So suppose $\mathcal{F},\mathcal{G}$ are $\mathrm{I}\neq$. Put

$$\mathcal{F}^{(1)}=\{X\in\mathcal{F}: \exists Y\in\mathcal{G}, Y\subset X\},\quad \mathcal{F}^{(2)}=\mathcal{F}\setminus\mathcal{F}^{(1)},$$

$$\mathcal{G}^{(1)}=\{Y\in\mathcal{G}: \exists X\in\mathcal{F}, X\subset Y\},\quad \mathcal{G}^{(2)}=\mathcal{G}\setminus\mathcal{G}^{(1)}.$$

Since condition $\neq$ holds we have $\mathcal{F}\cap\mathcal{G}=\varnothing$ and

$$|\mathcal{F}\cup\mathcal{G}|=|\mathcal{F}^{(1)}|+|\mathcal{F}^{(2)}|+|\mathcal{G}^{(1)}|+|\mathcal{G}^{(2)}|.$$

It follows from the definitions that $\mathcal{F}^{(2)}\subset\mathrm{incomp}(\mathcal{G}^{(2)})$ so

$$\sqrt{|\mathcal{F}^{(2)}|}+\sqrt{|\mathcal{G}^{(2)}|}\leq\sqrt{2^n}.$$

Next let $\mathcal{H}=\{N\setminus X: X\in\mathcal{F}^{(1)}\cup\mathcal{G}^{(1)}\}$ so

$$|\mathcal{H}|=|\mathcal{F}^{(1)}|+|\mathcal{G}^{(1)}|.$$

Assume that $Z\in\mathcal{F}\cap\mathcal{H}$. Then $Z=N\setminus X$ with either $X\in\mathcal{F}^{(1)}$ or $X\in\mathcal{G}^{(1)}$. In the first case there is a $Y\in\mathcal{G}$ with $Y\subset X$ and hence $Z\cap Y=\varnothing$, contradicting I. In the second case $Z\cap X=\varnothing$, again contradicting I. This shows that $\mathcal{F}\cap\mathcal{H}=\varnothing$ and by symmetry $(\mathcal{F}\cup\mathcal{G})\cap\mathcal{H}=\varnothing$. Let us define real numbers

$$a_1=|\mathcal{F}^{(1)}|/2^n,\qquad a_2=|\mathcal{F}^{(2)}|/2^n,\qquad b_1=|\mathcal{G}^{(1)}|/2^n,\qquad b_2=|\mathcal{G}^{(2)}|/2^n.$$

Then since $|\mathcal{F}\cup\mathcal{G}\cup\mathcal{H}|\leq 2^n$ the inequality $\mu(n)\leq(3-\sqrt{5})/2$ comes from

LEMMA 8.1 *Over reals $a_1, a_2, b_1, b_2\geq 0$ satisfying*

$$\sqrt{a_2}+\sqrt{b_2}\leq 1, \tag{8.1}$$

and

$$2a_1+a_2+2b_1+b_2\leq 1, \tag{8.2}$$

we have

$$\max\{\min\{a_1+a_2, b_1+b_2\}\}=(3-\sqrt{5})/2,$$

with equality if and only if, possibly by exchanging a and b,

$$0=b_1<a_2=(7-3\sqrt{5})/2<a_1=-2+\sqrt{5}<b_2=a_1+a_2=(3-\sqrt{5})/2.$$

PROOF Let a_1, a_2, b_1, b_2 be chosen to maximize the minimum. Without loss of generality we can assume $a_1+a_2=b_1+b_2$. Let $x=\min\{a_2, b_1\}$. Then by squaring twice we can verify that

$$\sqrt{(a_2-x)}+\sqrt{(b_2+x)}\leqslant\sqrt{a_2}+\sqrt{b_2}.$$

We change variables by adding x to a_1, b_2 and subtracting x from a_2, b_1. Clearly (8.1), (8.2) still hold and two cases arise.

Case $a_2=0$ Here (8.2) implies (8.1). So we want $\max\{b_1+b_2\}$ subject to $4b_1+3b_2\leqslant 1$. This max is clearly $\frac{1}{3}$ and $\frac{1}{3}<(3-\sqrt{5})/2$.

Case $b_1=0$ Here we want $\max\{a_1+a_2\}$ subject to $\sqrt{a_2}+\sqrt{(a_1+a_2)}\leqslant 1$ and $3a_1+2a_2\leqslant 1$. Put $y=\min\{1-3a_1-2a_2, a_2\}$ so $y\geqslant 0$. We change variables by adding y to a_1 and subtracting y from a_2. Then we assume $3a_1+2a_2=1$, for otherwise we are back in the case $a_2=0$. Eliminating a_2 we now want $\max\{(1-a_1)/2\}$ subject to

$$\sqrt{\{(1-3a_1)/2\}}+\sqrt{\{(1-a_1)/2\}}\leqslant 1.$$

Squaring this inequality twice and solving a quadratic equation shows that the best value of a_1 is $-2+\sqrt{5}$ and Lemma 8.1 follows.

9. To find $\lambda(n)$, $\mu(n)$ when Γ is a directed edge

We continue using the notation of Section 8 but now the edge $(1, 2)$ is directed. We assume that $C\in J$ because the other cases are covered by our work on undirected Γ.

THEOREM 9.1 *When Γ is a directed edge*

$$\lambda(n)=\begin{cases}1+2^{-n}\,\mathrm{Sper}(n) & \text{if } J=C,\\ 1 & \text{otherwise.}\end{cases}$$

PROOF Two cases must be considered:

Case $J=C$ Since $\mathscr{F}\cap\mathscr{G}$ is an antichain we have $|\mathscr{F}|+|\mathscr{G}|=|\mathscr{F}\cup\mathscr{G}|+|\mathscr{F}\cap\mathscr{G}|\leqslant 2^n+\mathrm{Sper}(n)$. There is equality if $\mathscr{F}=\{X\subset N: \lfloor\frac{1}{2}n\rfloor\leqslant|X|\}$ and $\mathscr{G}=\{X\subset N: |X|\leqslant\lfloor\frac{1}{2}n\rfloor\}$.

Case $J\neq C$ We have $1\leqslant\lambda(n)$ by $\mathscr{F}=2^n$ and $\mathscr{G}=\varnothing$. Ignoring the condition C we see by Theorem 8.1 that $\lambda(n)\leqslant 1$. □

THEOREM 9.2 *When Γ is a directed edge*

$$\begin{array}{lll}
\mu(n)=\frac{1}{2}\{1+2^{-n}\,\mathrm{Sper}(n)\} & \textit{if} & n=2m \textit{ and } J=C,\\
\mu(n)=\frac{1}{2}\left\{1+2^{-2m}\binom{2m}{m-1}\right\} & \textit{if} & n=2m+1 \textit{ and } J=C,\\
\mu(n)=\frac{1}{2} & \textit{if} & J=C\neq,\\
\mu=(3-\sqrt{5})/2 & \textit{for} & J=IC, UC,\\
\mu(n)\leqslant\mu=(3-\sqrt{5})/2 & \textit{for} & J=IC\neq, UC\neq,\\
\mu(n)\leqslant\mu=\frac{1}{4} & \textit{for} & J=IUC, IUC\neq.
\end{array}$$

PROOF Again a number of cases must be considered:

Case $J=C$ When n is even the result follows from Theorem 9.1. The authors conjectured the result for n odd but it seems to be much deeper than n even. It is the main result in a paper of Daykin **[4]**, who does not celebrate birthdays but wishes his friend Paul everlasting life in happiness.

Case $J=IC$ If $\mathscr{H}=\mathscr{F}\cap\mathscr{G}$ then $\mathscr{H}$ is an antichain. Let $\mathscr{F}'=\mathscr{F}\backslash\mathscr{H}$ and $\mathscr{G}'=\mathscr{G}\backslash\mathscr{H}$. Then $\mathscr{F}', \mathscr{G}'$ have properties I$\neq$ so $\min\{|\mathscr{F}'|, |\mathscr{G}'|\}\leqslant 2^n(3-\sqrt{5})/2$ by Theorem 8.2. It follows that

$$\mu(n)\leqslant(3-\sqrt{5})/2+2^{-n}\,\mathrm{Sper}(n).$$

The result now follows by Example 7.1. The example $\mathscr{F}=\{X\subset N: 7\leqslant|X|\}$ and $\mathscr{G}=\{X\subset N: 6\leqslant|X|\leqslant 7\}$ with $n=12$ shows that $\mu(12)\geqslant 0.3872>0.381\,97=(3-\sqrt{5})/2$.

Remaining cases These follow from Theorem 8.2 by deleting condition C.

10. To find $\mu(n)$ when $v>2$ and Γ is undirected

Let $\mathscr{H}$ be a set $\{H_1, \ldots, H_m\}$ of distinct finite sets H_i so $\mathscr{H}$ is a hypergraph. We recall that the fractional chromatic number $\chi^*(\mathscr{H})$ is defined as

$$\chi^*(\mathscr{H})=\min\left\{\sum_{1\leqslant i\leqslant m} x(H_i)\right\} \tag{10.1}$$

evaluated over all real numbers $x(H_1), \ldots, x(H_m)\geqslant 0$ such that

$$1\leqslant\sum_{\substack{1\leqslant i\leqslant m\\ p\in H_i}} x(H_i)\quad\text{for all}\quad p\in H_1\cup\ldots\cup H_m. \tag{10.2}$$

We need χ^* for

THEOREM 10.1 *Let $\mathscr{H}$ be the set of all distinct, maximal by inclusion,* *independent subsets of* $V(\Gamma)$. *If* Γ *is undirected and* J *is* $\neq$ *then*

$$\mu = 1/\chi^*(\mathcal{H})$$

and

$$\mu - 2^{-n}\,\mathrm{Sper}(v) \leq \mu(n) \leq \mu.$$

PROOF (i) Let $\mathcal{F}_1, \ldots, \mathcal{F}_v$ satisfy $J = \neq$. In this part we show that if $\delta = \min\{|\mathcal{F}_i|\}$ then $\delta \leq 1/\chi^*$. If $\delta = 0$ we have nothing to do so we assume $\delta > 0$. Let $M_1, \ldots, M_{2^n}$ be the distinct subsets of N. Put

$$A_j = \{k: 1 \leq k \leq v, M_j \in \mathcal{F}_k\} \quad \text{for} \quad 1 \leq j \leq 2^n,$$

so A_j is an independent subset of $V(\Gamma)$. Let $H_1, \ldots, H_m$ be the distinct members of $\mathcal{H}$. Choose any map $\omega: \{1, \ldots, 2^n\} \to \{1, \ldots, m\}$ such that $A_j \subset H_{\omega(j)}$ for all j. For $1 \leq i \leq m$ let ν_i be the number of times that H_i occurs in the list $H_{\omega(1)}, \ldots, H_{\omega(2^n)}$, and give to H_i the weight $x(H_i) = \nu_i/\delta \geq 0$.

Now $p \in H_1 \cup \ldots \cup H_m = V(\Gamma)$ means that $1 \leq p \leq v$, and for each such p we have

$$\begin{aligned} 1 \leq \delta^{-1}|\mathcal{F}_p| &= \delta^{-1} \sum_{\substack{1 \leq j \leq 2^n \\ p \in A_j}} 1 \\ &\leq \delta^{-1} \sum_{\substack{1 \leq j \leq 2^n \\ p \in H_{\omega(j)}}} 1 \\ &= \delta^{-1} \sum_{\substack{1 \leq i \leq m \\ p \in H_i}} \Bigg(\sum_{\substack{1 \leq j \leq 2^n \\ H_i = H_{\omega(j)}}} 1 \Bigg) \\ &= \delta^{-1} \sum_{\substack{1 \leq i \leq m \\ p \in H_i}} \nu_i = \sum_{\substack{1 \leq i \leq m \\ p \in H_i}} x(H_i). \end{aligned}$$

So these weights satisfy the conditions (10.2). Hence

$$\chi^* \leq \sum_{1 \leq i \leq m} x(H_i) = \delta^{-1} \sum_{1 \leq i \leq m} \nu_i = \delta^{-1} 2^n,$$

giving $\mu(n) \leq 1/\chi^*$ as required.

(ii) We here show that $(1/\chi^*) - 2^{-n}\,\mathrm{Sper}(v) \leq \mu(n)$. Now $\mathcal{H} = \{H_1, \ldots, H_m\}$ is an antichain on $V(\Gamma)$, so $m \leq \mathrm{Sper}(v)$. Choose any solution x of the $\chi^*(\mathcal{H})$ problem, this means that

$$\chi^*(\mathcal{H}) = \sum_{1 \leq i \leq m} x(H_i)$$

and (10.2) holds. For $1 \leq i \leq m$ put

$$d_i = \lfloor 2^n x(H_i)/\chi^* \rfloor,$$

so

$$\sum d_i \leq \sum 2^n x(H_i)/\chi^* = 2^n.$$

Let $\mathcal{G}_1, \ldots, \mathcal{G}_m$ be pairwise disjoint sets of subsets of N with $d_i = |\mathcal{G}_i|$ for $1 \leq i \leq m$. Then put

$$\mathcal{F}_p = \bigcup_{\substack{1 \leq i \leq m \\ p \in H_i}} \mathcal{G}_i \quad \text{for} \quad 1 \leq p \leq v.$$

Then for each p, by using (10.2), we have

$$|\mathcal{F}_p| = \sum_{\substack{1 \leq i \leq m \\ p \in H_i}} d_i \geq -m + \sum_{\substack{1 \leq i \leq m \\ p \in H_i}} 2^n x(H_i)/\chi^* \geq -m + 2^n/\chi^*,$$

and the inequality for $\mu(n)$ follows. As $n \to \infty$ we have $2^{-n}\,\mathrm{Sper}(v) \to 0$ so $\mu = 1/\chi^*$.

Finally we must verify that $\mathcal{F}_1, \ldots, \mathcal{F}_v$ satisfy the condition $\neq$. So suppose that (i, j) is an edge and $X \in \mathcal{F}_i \cap \mathcal{F}_j$. Since $\mathcal{G}_1, \ldots, \mathcal{G}_m$ are disjoint, there is a unique k with $X \in \mathcal{G}_k$. Hence $\mathcal{G}_k \subset \mathcal{F}_i \cap \mathcal{F}_j$ and this implies that $i, j \in H_k$, contradicting the fact that H_k is an independent set. This completes the proof of Theorem 10.1. □

REMARK 10.1 An obvious candidate for an undirected Γ is the complete graph K_v. Even for this graph and $J = \mathrm{C}$ we do not know $\mu(n)$. When $v = \mathrm{Sper}(m)$ for some m an obvious example shows that $1/2^m \leq \mu(n)$ for $m \leq n$. The best example we could find for $v = 4$ has $\mu(n) = \frac{5}{64}$. For this example let Q_1, Q_2, Q_3, Q_4 be a partition of the 20 subsets of $\{1, \ldots, 6\}$ of cardinality 3 with each Q_i having five of the subsets. Then put $\mathcal{F}_i = \{X \subset N: X \cap \{1, \ldots, 6\} \in Q_i\}$ for $1 \leq i \leq 4$.

PROOF OF LEMMA 2.1 We must construct an example. Let Γ be the complete undirected graph on

$$v = \binom{2m-1}{m} \geq 3$$

vertices. Let $Y_1, \ldots, Y_v$ be the v members of $\{Y \subset \{2, 3, \ldots, 2m\}: |Y| = m\}$. For $1 \leq i \leq v$ let $Y_i^c = \{1, 2, \ldots, 2m\} \backslash Y_i$. Finally for $n \geq 2m$ and $1 \leq i \leq v$ put

$$\mathcal{F}_i = \{X \subset N: X \cap \{1, 2, \ldots, 2m\} \text{ is } Y_i \text{ or } Y_i^c\},$$

so $|\mathcal{F}_i| = 2^{n-2m+1}$. This example is IUC$\neq$ and the lemma follows. □

11. When Γ is directed

Here and elsewhere

$$\text{above } \mathscr{F} = \{X \subset N : \exists Y \in \mathscr{F}, Y \subset X\},$$
$$\text{below } \mathscr{F} = \{X \subset N : \exists Y \in \mathscr{F}, X \subset Y\}.$$

We need this result of great importance:

LEMMA 11.1 (Kleitman [**12**]) $|\mathscr{F}|\, 2^n \leq |\text{above } \mathscr{F}|\, |\text{below } \mathscr{F}|$.

THEOREM 11.1 *If $v \geq 3$ and Γ is a directed circuit and $J = C\neq$ then $\sum \sqrt{|\mathscr{F}_i|} \leq (v/2)\sqrt{2^n}$ and $\mu(n) = \frac{1}{4}$.*

PROOF The conditions are that if $X \in \mathscr{F}_i$ and $Y \in \mathscr{F}_{i+1}$ then $X \not\subset Y$ and $X \neq Y$. Subscripts are taken mod v. For $1 \leq i \leq v$ let $a_i = 2^{-n}\, |\text{above } \mathscr{F}_i|$, $b_i = 2^{-n}\, |\text{below } \mathscr{F}_i|$. Since $J = C\neq$ we have $(\text{above } \mathscr{F}_i) \cap (\text{below } \mathscr{F}_{i+1}) = \varnothing$ so $a_i + b_{i+1} \leq 1$. Consequently, summing over i, we get $\sum (a_i + b_i) \leq v$. Now Lemma 11.1 says that

$$2^{-n}\, |\mathscr{F}_i| \leq a_i b_i \leq \{(a_i + b_i)/2\}^2.$$

We take the square root and sum to get our first result. For the second result, $\mu(n) = \frac{1}{4}$, notice that since $\sum (a_i + b_i) \leq v$ there is an i with $a_i + b_i \leq 1$. This implies that $a_i b_i \leq \frac{1}{4}$ so $2^{-n}\, |\mathscr{F}_i| \leq \frac{1}{4}$. Then examples to show the theorem is best possible are, for n even $\mathscr{G}_{12}, \mathscr{G}_{21}, \mathscr{G}_{12}, \mathscr{G}_{21}, \ldots$, and for n odd $\mathscr{G}_{12}, \mathscr{G}_{31}, \mathscr{G}_{23}, \mathscr{G}_{12}, \mathscr{G}_{21}, \ldots$. □

REMARK 11.1 We think that Theorem 11.1 holds for $J = C$ if n is large.

THEOREM 11.2 *If Γ is the directed path $(1, 2), (2, 3)$ and $J = C\neq$ then*

$$\sqrt{|\mathscr{F}_2|} + \sqrt{(|\mathscr{F}_1| + |\mathscr{F}_3|)} \leq (\tfrac{3}{2})\sqrt{2^n}.$$

PROOF Without loss of generality we may assume $\mathscr{F}_1 = \text{above } \mathscr{F}_1$, $\mathscr{F}_3 = \text{below } \mathscr{F}_3$ and $\mathscr{F}_2 \subset (2^n \backslash \mathscr{F}_1) \cap (2^n \backslash \mathscr{F}_3)$. Put $a = 2^{-n}\, |\mathscr{F}_1|$, $b = 2^{-n}\, |\mathscr{F}_2|$, $c = 2^{-n}\, |\mathscr{F}_3|$ then Lemma 11.1 says that $b \leq (1-a)(1-c)$. Hence if $d = (a+c)/2$ then

$$\sqrt{b} + \sqrt{(a+c)} \leq \sqrt{(2d)} + \sqrt{\{(1-a)(1-c)\}} \leq \sqrt{(2d)} + (1-d) \leq \tfrac{3}{2}.$$

The example $\mathscr{F}_1 = \mathscr{G}_{1,0}$, $\mathscr{F}_2 = \mathscr{G}_{2,1}$, $\mathscr{F}_3 = \mathscr{G}_{0,2}$ shows that the theorem is best possible. □

REMARK 11.2 We now mention a result of a different kind. Let

$\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3, \mathscr{F}_4$ be such that if $1 \leq i \leq 4$ and $X \in \mathscr{F}_i$, $Y \in \mathscr{F}_{i+1}$, $Z \in \mathscr{F}_{i+2}$ then we do not have either $X \subset Y \subset Z$ or $Y \subset X$, where suffices are taken mod 4. Then Hilton [**11**] has proved that $\sum \sqrt{|\mathscr{F}_i|} \leq 2(\sqrt{2^n})$.

12. Further problems

The KLYM poset P is log convex if $\nu(i-1)\nu(i+1) \leq \nu(i)\nu(i)$ for $0 < i < h$. Harper [**10**] proved that the direct product of such KLYM posets forms a KLYM poset. It would be interesting to rework this paper with 2^n replaced by P^n for such a P. If P was also a distributive lattice then so too would be P^n. Lemmas 11.1 and 6.1 of Kleitman and Seymour which we used hold in distributive lattices [**1, 2**].

The number $(3-\sqrt{5})/2$ does not appear to play a role in distributive lattices. This can be seen as follows. First note that the proofs of the upper bounds for the cases $J = \neq, C, C\neq$ of Theorem 8.2 are valid for distributive lattices. Then secondly consider the two examples below in the (distributive) lattice of divisors of the integer $2^r 3^r$.

EXAMPLE 12.1 Let $F = \{2^s 3^t : 0 < s < t < r\}$ and $G = \{2^s 3^t : 0 < t < s < r\}$. Then we get properties corresponding to $IU\neq$ and $\frac{1}{2} \leq \mu(n)$.

EXAMPLE 12.2 Let $F = \{2^s 3^t : 0 < s < \frac{1}{2}r < t < r\}$ and $G = \{2^s 3^t : 0 < t < \frac{1}{2}r < s < r\}$. Then we get properties like $IUC\neq$ and $\frac{1}{4} \leq \mu(n)$.

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10

COMPLETION OF SPARSE PARTIAL LATIN SQUARES

David E. Daykin and Roland Häggkvist

ABSTRACT Let P be a partial $n\times n$ latin square with $0, 1, \ldots, n-1$ as symbols. (1) If $n=16k$ and each row, column and symbol is used at most $(\sqrt{k})/32$ times then P can be completed. (2) Form a partial $mn\times mn$ latin square Q from P as follows. Replace each cell c of P by an $m\times m$ array $A(c)$. If c is empty then $A(c)$ is empty. If c has symbol x then exactly one cell of $A(c)$ is filled from among $mx, mx+1, \ldots, mx+m-1$ in any way. We conjecture that Q can be completed for $m\geq 2$ and prove it for $m\equiv 0 \pmod{16}$.

1. Introduction

A partial $n\times n$ latin square P is an $n\times n$ array where some cells are filled with one of the symbols $0, 1, \ldots, n-1$ in such a way that no symbol occurs twice in a row or column. If every cell is filled then P is a latin square. If every empty cell can be filled so that the result is a latin square then we say that P can be completed. Our main result, which we think to be the first of its kind, is in the style of Evan's problems.

PROPOSITION 1 *If P is a partial $16k\times 16k$ latin square where each row, column and symbol is used at most $(\sqrt{k})/32$ times then P can be completed.*

The word sparse is in our title because $(\sqrt{k})/32$ is small. We think that the proposition would still be true if $16k$ and $(\sqrt{k})/32$ were replaced by k and uk respectively, where u is some constant, maybe $u=\frac{1}{4}$. A famous result of Ryser is

PROPOSITION 2 *If P is a partial $n\times n$ latin square whose filled cells consist of all the cells in an $r\times s$ rectangle then P can be completed if and only if each symbol occurs at least $r+s-n$ times in P.*

This result trivially implies that if $n=2d-1$ (respectively $n=2d$) and P has all its filled cells in the first $d-1$ (respectively d) rows and the first d columns then P can be completed. From this observation we easily got

PROPOSITION 3 *Any partial $n\times n$ latin square P can be partitioned into*

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