A new proof of Viazovska’s modular form inequalities for sphere packing in dimension 8

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Computer Algebra Workshop + Séminaire Philippe Flajolet
Institut Henri Poincaré

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Talk outline

1. Background: sphere packings in $\mathbb{R}^d$
2. Viazovska's solution of the sphere packing problem in dimension 8
3. Viazovska's modular form inequalities
4. A new proof

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Some useful references

Source material for this talk:

• My paper "On Viazovska's modular form inequalities" (PNAS, 2023).
• Chapter 6 + Appendix of my book "Topics in Complex Analysis" https://www.math.ucdavis.edu/~romik/topics-in-complex-analysis/
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Background: sphere packings in $\mathbb{R}^d$

The sphere packing problem in $\mathbb{R}^d$ asks: what is the densest way to pack unit spheres in $d$-dimensional space?

- **Trivial case:** $d = 1$.
- **(Relatively) easy case:** $d = 2$.
  
  The optimal circle packing is the hexagonal lattice packing, with packing density $\pi \frac{2}{\sqrt{3}}$.
  
  Proved by Gauss (1831), Thue (1890), Tóth (1941).

- **Famous case:** $d = 3$.
  
  Kepler’s conjecture from 1611 stated that the optimal density for sphere packing is $\pi \frac{3}{\sqrt{2}}$, achieved by the cubic close packing and the hexagonal close packing.
  
  Proved by Thomas Hales in 1998.

- The case $d = 8$.
  
  Maryna Viazovska proved in 2016 that for $d = 8$, the densest packing is the $E_8$ lattice packing, with packing density $\pi \frac{4}{384}$.

- The case $d = 24$.
  
  Viazovska with Cohn, Kumar, Miller, and Radchenko then proved that for $d = 24$, the densest packing is the Leech lattice packing, with packing density $\pi \frac{12}{12!}$.
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For $d = 2$, the optimal circle packing is the hexagonal lattice packing, with packing density $\frac{\pi}{2\sqrt{3}}$. Proved by Gauss (1831), Thue (1890), Tóth (1941).

For $d = 3$, Kepler's conjecture from 1611 stated that the optimal density for sphere packing is $\frac{\pi}{\sqrt{18}}$, achieved by the cubic close packing and the hexagonal close packing. Proved by Thomas Hales in 1998.

For $d = 8$, Maryna Viazovska proved in 2016 that the densest packing is the $E_8$ lattice packing, with packing density $\frac{\pi}{384}$. 

For $d = 24$, Viazovska with Cohn, Kumar, Miller, and Radchenko then proved that the densest packing is the Leech lattice packing, with packing density $\frac{\pi}{192 \cdot 12!}$. 

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In other dimensions the problem remains open.
The optimal lattices for sphere packing in $\mathbb{R}^d$ (continued)
Viazovska’s proof in dimension 8

It had previously been conjectured that the $E_8$ lattice packing, with packing density $\frac{\pi}{4} \frac{384}{3}$, is optimal. This gives a lower bound on the optimal packing density; Viazovska proved a matching upper bound, solving the problem.

Viazovska made use of a remarkable theorem from 2001, the Cohn-Elkies linear programming bounds. It reduced the problem to finding a magic function, an analytic object with certain properties.

Viazovska’s proof is complex-analytic. She used modular forms to construct the magic function for dimension 8. An extension of the method works for dimension 24.

One component of the proof makes extensive use of computer calculations.
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The Cohn-Elkies linear programming bounds

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a function and $\rho > 0$ a number. Assume that the following conditions are satisfied:

1. $f$ is a Schwartz function
2. $f(0) = b_f(0) > 0$ ($b_f$ is the Fourier transform of $f$)
3. $f(x) \leq 0$ for all $x \in \mathbb{R}^d$ such that $\|x\| \geq \rho$
4. $b_f(x) \geq 0$ for all $x \in \mathbb{R}^d$

Then the optimal packing density $\delta_d$ in $\mathbb{R}^d$ satisfies

$$\delta_d \leq \frac{\text{vol}(B_{\rho/2}(0))}{\rho^d} \times \text{[vol. of unit ball]}$$

The proof is an application of the Poisson summation formula from harmonic analysis; see the appendix of my book.

For the case $d = 8$, the sharp bound $\pi^{4/384}$ is obtained when $\rho = \sqrt{2}$. A function satisfying the conditions of the theorem for that $\rho$ is called a magic function.
The Cohn-Elkies linear programming bounds

Theorem (Cohn-Elkies, 2001)
Let \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) be a function and \( \rho > 0 \) a number. Assume that the following conditions are satisfied:

1. \( f \) is a Schwartz function
2. \( f(0) = b \)
3. \( f(x) \leq 0 \) for all \( x \in \mathbb{R}^d \) such that \( \|x\| \geq \rho \)
4. \( b f(x) \geq 0 \) for all \( x \in \mathbb{R}^d \)

Then the optimal packing density \( \delta_d \) in \( \mathbb{R}^d \) satisfies
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Then the optimal packing density \( \delta_d \) in \( \mathbb{R}^d \) satisfies

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\delta_d \leq \text{vol}(B_{\rho/2}(0)) = \frac{\rho^d}{2^d} \times [\text{vol. of unit ball}]
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Then the optimal packing density $\delta_d$ in $\mathbb{R}^d$ satisfies

$$\delta_d \leq \text{vol}(B_{\rho/2}(0)) = \frac{\rho^d}{2d} \times \text{[vol. of unit ball]}$$

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- For the case $d = 8$, the sharp bound $\frac{\pi^4}{384}$ is obtained when $\rho = \sqrt{2}$. A function satisfying the conditions of the theorem for that $\rho$ is called a magic function.
Applying the Cohn-Elkies bounds in practice
Cohn and Elkies applied their bound to numerically optimized bounding functions $f$, obtaining the best known (at the time) upper bounds for the sphere packing density in dimensions 4–36.
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In dimensions 2, 8 and 24, their bounds came extremely close to matching the known lower bounds.

<table>
<thead>
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<th>Dimension</th>
<th>log(density)</th>
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<tr>
<td>32</td>
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<td>36</td>
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They conjectured that in those dimensions there exists a “magic function” \( f \) certifying a \textit{sharp} bound.
Viazovska’s function \( \varphi : \mathbb{R}^8 \to \mathbb{R} \) is defined by

\[
\varphi(x) = -4 \sin^2 \pi \|x\|^2 \times \int_0^{\infty} e^{-\pi t \|x\|^2} \left( 108 (itE_4'(it) + 4E_4(it)) \right)^2 E_4(it)^3 - E_6(it)^2 + 128 \theta_3(it)^4 + \theta_4(it)^4 - \theta_2(it)^4 \theta_3(it)^8 + \theta_4(it)^4 - \theta_2(it)^4 \theta_3(it)^8 \right) dt,
\]

where \( E_4, E_6 \) are the Eisenstein series and \( \theta_2, \theta_3, \theta_4 \) are the Jacobi thetanull functions, defined by

\[
E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n}, \quad \theta_2(z) = \sum_{n=-\infty}^{\infty} q^{n+1/2}, \quad E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^{2n}, \quad \theta_3(z) = \sum_{n=-\infty}^{\infty} q^n, \quad \theta_4(z) = \sum_{n=-\infty}^{\infty} (-1)^n q^n,
\]

(with the standard notation \( q = e^{\pi i z}, \sigma_\alpha(n) = \sum_{d \mid n} \alpha(d) \)).
Viazovska’s magic function

Viazovska’s function $\varphi : \mathbb{R}^8 \to \mathbb{R}$ is defined by (the analytic continuation of)

$$\varphi(x) = -4 \sin^2 \left( \frac{\pi \|x\|^2}{2} \right) \times \int_0^\infty e^{-\pi t \|x\|^2} \left[ 108 \frac{(itE'_4(it) + 4E_4(it))^2}{E_4(it)^3 - E_6(it)^2} + 128 \left( \frac{\theta_3(it)^4 + \theta_4(it)^4}{\theta_2(it)^8} + \frac{\theta_4(it)^4 - \theta_2(it)^4}{\theta_3(it)^8} \right) \right] dt,$$

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E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^{2n}, \quad \theta_2(z) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}, \\
E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^{2n}, \quad \theta_3(z) = \sum_{n=-\infty}^{\infty} q^{n^2}, \\
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(with the standard notation \( q = e^{\pi iz}, \sigma_\alpha(n) = \sum_{d \mid n} d^\alpha \)).
Theorem (Viazovska)

\( \varphi \) is a magic function for dimension 8. More precisely, it has the following properties:

1. \( \varphi \) is a Schwartz function
2. \( \varphi(0) = 240 \pi \)
3. \( \varphi(x) \geq 0 \) for all \( x \in \mathbb{R}^8 \)
4. \( \varphi(x) \leq 0 \) for all \( x \in \mathbb{R}^8 \) with \( \|x\| \geq \sqrt{2} \)

Using the Cohn-Elkies linear programming bound, the above properties imply that \( \varphi \) certifies an upper bound of \( \frac{\pi}{4} \cdot \frac{4}{384} \) for sphere packing density in \( \mathbb{R}^8 \). This matches the packing density of the \( \mathcal{E}_8 \) lattice packing.

It remains to prove the claimed properties. This is not trivial. (Related, and much more nontrivial: the reasoning that led to the strange formula for \( \varphi \).)
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φ is a magic function for dimension 8. More precisely, it has the following properties:

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1. φ is a Schwartz function
2. φ(0) = 240π, \hat{φ}(0) = 240π (\hat{φ} = the Fourier transform of φ)
3. \|x\| ≥ \sqrt{2} implies φ(x) ≥ 0
4. φ(x) ≤ 0 for all x ∈ \mathbb{R}^8 with \|x\| ≥ \sqrt{2}

Using the Cohn-Elkies linear programming bound, the above properties imply that φ certifies an upper bound of \(\pi \frac{4}{384}\) for sphere packing density in \(\mathbb{R}^8\). This matches the packing density of the \(E_8\) lattice packing.

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φ is a magic function for dimension 8. More precisely, it has the following properties:

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2. φ(0) = 240π = 4π
3. φ(x) ≥ 0 for all x ∈ ℝ^8
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The modular forms in the definition of $\varphi$

The problem boils down to understanding the properties of the modular forms in the definition of $\varphi$. Let $\mathbb{H}$ denote the upper half plane. Define functions $U : \mathbb{H} \to \mathbb{C}$, $V : \mathbb{H} \to \mathbb{C}$ by

$$U(z) = 108 \frac{(zE_4'(z) + 4E_4(z))^2}{E_4(z)^3 - E_6(z)^2}$$

$$V(z) = 128 \left( \frac{\theta_3(z)^4 + \theta_4(z)^4}{\theta_2(z)^8} + \frac{\theta_4(z)^4 - \theta_2(z)^4}{\theta_3(z)^8} \right).$$
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Define functions $\varphi_\pm : \mathbb{R}^8 \to \mathbb{R}$ by (the analytic continuation of)

$$\varphi_+(x) = -4 \sin^2 \left( \frac{\pi \|x\|^2}{2} \right) \int_0^\infty e^{-\pi t\|x\|^2} U(it) \, dt$$

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\]

so that \( \varphi = \varphi_+ + \varphi_- \).
The definitions of $U(z)$, $V(z)$ were carefully chosen to satisfy several conditions, including, crucially,

$$\hat{\varphi}_+ = \varphi_+, \quad \hat{\varphi}_- = -\varphi_-.$$

(That is, $\varphi_\pm$ are eigenfunctions of the Fourier transform with resp. eigenvalues $\pm 1$.)
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The inequalities for $\varphi$ and $\hat{\varphi}$ will therefore follow from the following result:
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**Theorem (Viazovska)**

The functions $U, V$ satisfy the inequalities

\begin{align*}
U(it) + V(it) &\geq 0 \quad (t > 0) \quad (V1) \\
U(it) - V(it) &\leq 0 \quad (t > 0) \quad (V2)
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- The inequalities (V1)–(V2) seem unnatural, because they relate modular forms that belong to different modular form spaces. This makes it difficult to think of a conceptual reason why they should be true.
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Viazovska’s modular form inequalities (continued)

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A new proof

I will now show a new proof of (V1)–(V2) that does not rely on computer calculations.

Reminder (1). \( E_4, E_6, \theta_2, \theta_3, \theta_4 \) are modular forms satisfying the modular transformation properties:

\[
E_4(z + 1) = E_4(z), \quad E_4(-1/z) = z^4 E_4(z),
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\[
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\]

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Reminder (2). \( E_3^4 - E_2^6 \) is a scaling of the modular discriminant:

\[
E_3^4 - E_2^6 = 1728 \left(\frac{2\pi}{\lambda}\right)^{12} \Delta(z) = 1728 q^{-2} \sum_{n=1}^{\infty} (1 - q^n)^24 = 27 \left(\frac{\theta_2 \theta_3 \theta_4}{\lambda}\right)^8.
\]

Reminder (3). The modular lambda function \( \lambda = \frac{\theta_4}{\lambda} \) is \( \lambda = 1 - \theta_4^4 \theta_3^3 \). For \( t > 0 \), \( \lambda(it) \in (0, 1) \).
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A new proof of (V1)–(V2), part I: proof of (V1)

First, observe that $U(it) \geq 0$ for all $t > 0$ since, by inspection of the relevant Fourier series, we have $itE_4'(it) + 4E_4(it) \in \mathbb{R}$, and separately we have $E_4(z) - E_6(z)^2 > 0$ for $t > 0$ by the infinite product formula from the previous slide.

Similarly, it also holds that $V(it) \geq 0$ for $t > 0$. To see this, rewrite $V(z)$ in terms of $\theta_3$ and the modular lambda function as $V = 128\theta_3^3 + \theta_4^4 \theta_8^2 + \theta_4^4 - \theta_4^2 \theta_8^3 = \ldots = 128\theta_4^3(1 - \lambda)(2 + \lambda + 2\lambda^2)\lambda^2$.

Then use the facts that $\theta_3(it) > 0$ (trivially), that $\lambda(it) \in (0, 1)$ for $t > 0$, and that the map $x \mapsto (1 - x)(2 + x + 2x^2)x^2$ takes positive values for $x \in (0, 1)$. 


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V = 128 \left( \frac{\theta_3^4 + \theta_2^4}{\theta_2^8} + \frac{\theta_4^4 - \theta_2^4}{\theta_3^8} \right) = \ldots = \frac{128}{\theta_3^4} \frac{(1 - \lambda)(2 + \lambda + 2\lambda^2)}{\lambda^2}.
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A new proof of (V1)–(V2), part I: proof of (V1)
A new proof of (V1)–(V2), part II: proof of (V2)

Step 1: A bit of cleanup

Define functions

\[ F(z) = 108(1 - E_4/3 - E_2/6) U(z) = (E'4/2 z^2 + 8 E_4 E'4 z + 16 E_2 E_4), \]

\[ e F(z) = 108(1 - E_4/3 - E_2/6) z^2 U(-1/z) = (E'4/2 z^2), \]

\[ G(z) = 108(1 - E_4/3 - E_2/6) V(z) = 8 \theta_2^2 (\theta_1^3 + \theta_4^2 \theta_8^3 + \theta_8^2 \theta_4^2 - \theta_1^4) \]

\[ e G(z) = 108(1 - E_4/3 - E_2/6) z^2 V(-1/z) = -8 \theta_2^2 (\theta_1^3 + \theta_4^2 \theta_8^3 + \theta_8^2 \theta_4^2 - \theta_1^4). \]

(making use of the modular transformation properties).

Trivially, the inequality (V2) is equivalent to the pair of inequalities

\[ -e F(it) < -e G(it) \quad (t \geq 1), \]

(V2-I)

\[ F(it) < G(it) \quad (t \geq 1). \]

(V2-II)
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Define functions

\[ F(z) = \frac{1}{108}(E_4^3 - E_6^2)U(z) = (E'_4)^2 z^2 + 8E_4E'_4 z + 16E_4^2, \]

\[ \tilde{F}(z) = \frac{1}{108}(E_4^3 - E_6^2)z^2 U(-1/z) = (E'_4)^2, \]

\[ G(z) = \frac{1}{108}(E_4^3 - E_6^2)V(z) = 8\theta_4^8(\theta_3^{12} + \theta_4^4\theta_3^8 + \theta_2^8\theta_4^4 - \theta_2^{12}), \]

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(making use of the modular transformation properties).

Trivially, the inequality (V2) is equivalent to the pair of inequalities

\[ -\tilde{F}(it) < -\tilde{G}(it) \quad (t \geq 1), \quad \text{(V2-I)} \]

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A new proof of (V1)–(V2), part II: proof of (V2) (cont’d)

Step 2: Understanding the behavior at $t = 1$
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Theorem (Gauss, Ramanujan, folklore)

We have the explicit evaluations

\[
E_4(i) = \frac{3\Gamma(1/4)^8}{64\pi^6}, \quad \theta_2(i) = \frac{\Gamma(1/4)}{(2\pi)^{3/4}},
\]

\[
E'_4(i) = \frac{3\Gamma(1/4)^8}{32\pi^6}i, \quad \theta_3(i) = \frac{\Gamma(1/4)}{\sqrt{2}\pi^{3/4}},
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(where \( \Gamma(\cdot) \) denotes the Euler gamma function).
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See p. 257 of my book for a proof sketch and references.
Step 3: Leveraging monotonicity

- $e_F(z) = 230400\pi^2 q^4 + 8294400\pi^2 q^6 + 113356800\pi^2 q^8 + 831283200\pi^2 q^{10} + 4337971200\pi^2 q^{12} + \ldots$

- $e_G(z) = 163840q^3 + 16121856q^5 + 333250560q^7 + 3199467520q^9 + 19472547840q^{11} + \ldots$

Note that $q^4 = e^{-4\pi t} \ll e^{-3\pi t} = q^3$ for large $t$, so the inequality $-e_F(it) < -e_G(it)$ holds asymptotically.

To prove the stronger claim that it holds for $t \geq 1$, note that the Fourier coefficients in both series are positive. This is easy to prove from the definitions. In particular, the function $t \mapsto -q^{-3}e_F(it)$ is a decreasing function of $t$, so that for $t \geq 1$,

$$-e^{3\pi t}e_F(it) \leq e^{3\pi}e_F(i) = -e^{3\pi}E_4'(i)^2 \approx 1050430.78.$$ 

This in turn is $< 163840$, which is a lower bound for $-e^{3\pi t}e_G(it)$. 
Step 3: Leveraging monotonicity

Proof of \( (V2-I) \). Observe that

\[
-\tilde{F}(z) = 230400\pi^2 q^4 + 8294400\pi^2 q^6 + 113356800\pi^2 q^8 \\
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Step 3: Leveraging monotonicity

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Step 3: Leveraging monotonicity

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\[
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Step 3: Leveraging monotonicity

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\[-e^{3\pi t}\tilde{F}(it) \leq e^{3\pi} \tilde{F}(i) = -e^{3\pi} E_4'(i)^2 = e^{3\pi} \frac{9\Gamma(1/4)^{16}}{1024\pi^{12}}\]

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Step 3: Leveraging monotonicity

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This in turn is < 163840, which is a lower bound for \(-e^{3\pi t} \tilde{G}(it)\).

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Summarizing this argument:
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Plots of $-\tilde{F}(it)$, $-\tilde{G}(it)$
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Plots of $-e^{3\pi t}\tilde{F}(it)$, $-e^{3\pi t}\tilde{G}(it)$
Proof of (V2-II). Imitating the approach for (V2-I), note that

\[ F(it) = 16 + (-3840\pi t + 7680)q^2 \]
\[ + (230400\pi^2 t^2 - 990720\pi t + 990720)q^4 \]
\[ + (8294400\pi^2 t^2 - 25205760\pi t + 16803840)q^6 + \ldots, \]

\[ G(it) = 16 + 1920q^2 - 81920q^3 + 1077120q^4 - 8060928q^5 \]
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Define renormalized functions

\[ K(z) = -\frac{F(z) - 16}{q^2} = -q^{-2}(E_4')^2z^2 - 8q^{-2}E_4'E_4z - 16q^{-2}(E_4^2 - 1), \]

\[ L(z) = -\frac{G(z) - 16}{q^2} = -8q^{-2} \left[ \theta_4^8(\theta_3^{12} + \theta_4^4\theta_3^8 + \theta_2^8\theta_4^4 - \theta_2^{12}) - 2 \right], \]
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\[ K(z) = -\frac{F(z) - 16}{q^2} = -q^{-2}(E'_4)^2 z^2 - 8q^{-2}E'_4E_4z - 16q^{-2}(E_4^2 - 1), \]
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The inequality (V2-II) is thus equivalent to the inequality

\[ K(it) > L(it) \quad (t \geq 1). \]
As in the earlier proof, we will bound each of $K(it)$ and $L(it)$ separately, obtaining the inequality (V2-II) from the combination of the following two lemmas:
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**Lemma (1)**

$L(it) \leq 2297$ for $t \geq 1$. 
As in the earlier proof, we will bound each of $K(it)$ and $L(it)$ separately, obtaining the inequality (V2-II) from the combination of the following two lemmas:

**Lemma (1)**

$L(it) \leq 2297$ for $t \geq 1$.

**Lemma (2)**

$K(it) \geq 3747$ for $t \geq 1$. 
Proof of Lemma (1). Again the idea is to leverage monotonicity.

Define $H(z) = L(z + 1) - L(z)^2 = \ldots = 4q^2 - \theta_8^2 (\theta_{12}^3 - \theta_{12}^4) + \theta_{12}^2 (\theta_8^3 + \theta_8^4)$.

Then for $t \geq 1$, 

$L(it) \leq -1920 + 81920q - 1077120q^2 + 8060928q^3 - 41725440q^4 + 166625280q^5 - 553054080q^6 + \ldots$

$\leq -1920 + H(it) - L(it)^2 \leq \ldots = -1920 + 3e^{2\pi\Gamma(1/4)^2/2048\pi^2} \approx 2296.16 \leq 2297$, which is what we wanted.
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\[ H(z) = \frac{L(z + 1) - L(z)}{2} = ... = 4q^{-2} (\theta_2^8(\theta_3^{12} - \theta_4^{12}) + \theta_2^{12}(\theta_3^8 + \theta_4^8)) \]

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$$W = \frac{1}{16}(6X^5 + 15X^4 Y + 10X^3 Y^2 + Y^5),$$ 

* This nonnegativity result was first proved by Slipper (2018), with a more complicated proof. See also [https://mathoverflow.net/q/441749/78525](https://mathoverflow.net/q/441749/78525).

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Proof of Lemma (2).

The asymptotic expansion of $K(it)$ is

$$K(it) = (3840\pi t - 7680) + \left(-\frac{230400}{\pi^2}t^2 + 990720\pi t - 990720\right)q^2 + \left(-\frac{8294400}{\pi^2}t^2 + 25205760\pi t - 16803840\right)q^4 + \ldots$$

With this motivation in mind, define

$$K_1(t) = 3840\pi t + \left(-\frac{230400}{\pi^2}t^2 + 990720\pi t - 990720\right)q^2,$$

$$K_2(t) = q^2E_4(it)^2 - 16q^2(E_4(it)^2 - 1) + \left(\frac{230400}{\pi^2}t^2 + 990720\right)q^2,$$

$$K_3(t) = -8iq^2E_4(it)E_4(it)t - (3840\pi t + 990720\pi q^2),$$

so that we have

$$K(it) = K_1(t) + K_2(t) + K_3(t).$$

The following claims are easy to check:

1. The function $K_1(t)$ is monotone increasing on $[1, \infty)$.
2. The function $K_2(t)$ is monotone increasing on $[1, \infty)$.
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Proof of Lemma (2). The asymptotic expansion of $K(it)$ is

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A new proof of (V1)--(V2), part II: proof of (V2) (cont’d)

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\[
K(it) = K_1(t) + K_2(t) + K_3(t) \geq K_1(t) + K_2(t) \geq K_1(1) + K_2(1) = -e^{2\pi} - E_4(i)^2 + 16E_4(i)^2 - 16 + 3840\pi + 9920 \approx 3747.1,
\]

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\approx 3747.1,
\]
Therefore, assuming \( t \geq 1 \),

\[
K(it) = K_1(t) + K_2(t) + K_3(t) \geq K_1(t) + K_2(t) \geq K_1(1) + K_2(1) \\
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as claimed.
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- We now understand it a bit better than before. But there seems more to understand still.
- Open problems:
  - Find a human proof of the analogous inequalities for the case of dimension 24.
  - Prove the analogous inequalities for dimensions that are multiples of 4. (Might require computer assistance?)
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